V4E2 - Numerical Simulation

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Exercise sheet 3.

To be handed in on **Tuesday, 08.05.2018**.

The next two exercises are devoted to the Hopf-Lax representation formula. Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be bounded and Lipschitz continuous. We call *Lagrangian* every convex function $L : \mathbb{R}^n \to \mathbb{R}$ satisfying the coercivity condition:

$$\lim_{|v|\to\infty} \frac{L(v)}{|v|} = +\infty$$

Define the function $u(x,t)$ by (a slight variant of) the Hopf-Lax formula:

$$u(x,t) := \min_{y\in\mathbb{R}^n} \Big\{ tL\Big(\frac{x-y}{t}\Big) + u_0(y) \Big\}$$

This function is continuous in the first variable.

Exercise 1. (A functional equality)

Prove that for each $x \in \mathbb{R}^n$ and $0 \leq s < t$, we have:

$$u(x,t) = \min_{y\in\mathbb{R}^n} \Big\{ (t-s)L\Big(\frac{x-y}{t-s}\Big) + u(y,s) \Big\}$$

(in other words, to compute $u(\cdot,t)$, we can calculate u at time s and then use $u(\cdot,s)$ as starting point in the remaining time interval $[s,t]$)
We give some hints for the inequality $\leq$:

- Fix $y \in \mathbb{R}^d$ and choose $z \in \mathbb{R}^n$ such that:
$$u(y,s) = sL\Big(\frac{y-z}{s}\Big) + u_0(z),$$
- use convexity with:
$$\frac{x-z}{t} = \Big(1 - \frac{s}{t}\Big)\frac{x-y}{t-s} + \frac{s}{t}\frac{y-z}{s},$$
- use continuity of $y \mapsto u(y,s)$.

(6 Punkte)

Recall the Legendre transform of L to be:

$$L^*(p) := \sup_{v\in\mathbb{R}^n} \{p \cdot v - L(v)\}$$

again a function $\mathbb{R}^d \to \mathbb{R}$. The corresponding Hamiltonian is then given by:

$$H := L^*$$

Exercise 2. (L and H are dual convex functions)

Prove that the following properties hold:

a) The mapping $p \to H(p)$ is convex;

b) it fulfills the coercivity condition

$$\lim_{|v|\to\infty} \frac{H(v)}{|v|} = +\infty,$$

c) $L = H^*$.

We give some hints for $L \leq H^*$:

$$H^*(v) = \sup_{p\in\mathbb{R}^n} \{p \cdot v - \sup_{r\in\mathbb{R}^n} \{p \cdot r - L(r)\}\}$$

convexity of L implies

$$\exists s \in \mathbb{R}^n : \quad L(r) \geq L(v) + s \cdot (r - v).$$

(6 Punkte)

Note that under the previous conditions, the Hopf-Lax formula for u can be rewritten with H^* giving exactly the equality needed for completing the proof of Theorem 14.
Let E be a closed subset of $\mathbb{R}^d$. The distance function $\mathbb{R}^d \to [0, \infty)$ is defined as

$$dist(x, E) \doteq min_{y\in E}|x - y|$$

Exercise 3. (A static Eikonal equation)

Let u be defined as the distance function from a closed subset E. Show that:

1 u is 1-Lipschitz, i.e. $|u(x) - u(y)| \leq |x - y|$;

2 u is the *unique* viscosity solution to the problem:

$$\begin{cases} |Du(x)| = 1 & x \in \mathbb{R}^d \setminus E \\ u = 0 & x \in E \end{cases}$$

(Hint: Use suitable change of coordinates and the uniqueness result from the lectures)

(6 Punkte)