V4E2 - Numerical Simulation

Sommersemester 2018
Prof. Dr. J. Garcke
Teaching assistant: Biagio Paparella
Tutor: Marko Rajković
(marko.rajkovic@uni-bonn.de)

UNIVERSITÄT BONN

Exercise sheet 4.

To be handed in on **Tuesday, 15.05.2018**.

Exercise 1. Prove the following inequalities, important (in particular) for the proof of Lemma 17:

(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$, which satisfies for a.e. $0 \leq t \leq T$ the differential inequality

$$\eta'(t) \leq \omega(t)\eta(t) + \psi(t)$$

where $\omega(t)$ and $\psi(t)$ are nonnegative, integrable functions on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \omega(s)ds}\left[\eta(0) + \int_0^t \psi(s)ds\right]$$

for all $0 \leq t \leq T$.

(ii) Let $\phi(\cdot)$ be a nonnegative, integrable function on $[0, T]$ which satisfies for a.e. $0 \leq t \leq T$ the integral inequality

$$\phi(t) \leq C_2 + \int_0^t C_1\phi(s)ds$$

for constants $C_1, C_2 > 0$. Then

$$\phi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$. Hints: (i): consider $\frac{d}{ds}(\eta(s)e^{-\int_0^s \omega(r)dr})$ (ii): Use (i) to prove (ii).

(6 Punkte)

Exercise 2. Given the initial data $y(t_0) = x_0$. We define the function:

$$h(t) := \int_{t_0}^t \ell(y_{x_0,t_0}(s), \alpha(s))ds + V(y_{x_0,t_0}(t), t).$$

where V is the usual value function, used for deriving (in class) an important equality called *minimum principle*.
Prove the following properties:

(i) h is nondecreasing for any control α,

(ii) h is constant if and only if the control α is optimal.

(4 Punkte)

Let E be a closed subset of $\mathbb{R}^d$. Recall that the distance function $\mathbb{R}^d \to [0, \infty)$ is defined to be:

$$dist(x, E) \doteq min_{y \in E} |x - y|$$

Exercise 3. (A time-dependent Eikonal equation)
Let $u_0 : \mathbb{R}^d \to \mathbb{R}$ be defined as:

$$u_0(x) = \begin{cases} 0 & x \in E \\ +\infty & x \notin E \end{cases}$$

Show that if the Hopf-Lax formula could be applied to the initial value problem:

$$\begin{cases} u_t + |Du|^2 = 0 & \mathbb{R}^d \times (0, \infty) \\ u = u_0 & \mathbb{R}^d \times \{t = 0\} \end{cases}$$

then it would give the solution:

$$u(x, t) = \frac{1}{4t} dist(x, E)^2$$

(6 Punkte)