V4E2 - Numerical Simulation

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Exercise sheet 5.

To be handed in on **Tuesday, 29.05.2018**.

In this problem set we develop an extension of the HBJ equation to the infinite-horizon setting and solve some optimal control problem with cost only at the end (i.e. $l = 0$). Finally, we check compatibility between two definitions of *consistency* for a numerical scheme.

Exercise 1. (Optimal control in infinite time setting)

Assume f and l satisfying our assumptions $(A5)$ and $(A6)$. Given a point $x \in \mathbb{R}^n$ and a control belonging to $\mathcal{A} \doteq \{\alpha : [0, \infty) \to A | \alpha(\cdot) \text{ is measurable } \}$, let $x(\cdot)$ be the unique solution to the ODE:

$$\begin{cases} \dot{x}(s) = f(x(s), \alpha(s)) & s > 0 \\ x(0) = x \end{cases}$$

Fix $\lambda > 0$ and define the *discounted cost*:

$$C_x[\alpha(\cdot)] \doteq \int_0^\infty e^{-\lambda s} l(x(s), \alpha(s)) ds$$

And so the *value function*:

$$V(x) \doteq \inf_{\alpha(\cdot) \in \mathcal{A}} C_x[\alpha(\cdot)]$$

Prove the following properties:

1 V is bounded and if $\lambda > Lip(f)$, then V is Lipschitz continuous (where $Lip(f) \doteq \sup_{x,y \in Dom(f), x \neq y} \frac{|f(x)-f(y)|}{|x-y|}$);

2 If $0 < \lambda \leq Lip(f)$, then V is Hoelder continuous for some exponent $0 < \beta < 1$;

3 The value function V is a viscosity solution of the PDE:

$$\lambda u - \min_{a \in A} \{ f(x, a) \cdot Du + l(x, a) \} = 0$$

in $\mathbb{R}^n$ (to clarify, if $v \in C^1(\mathbb{R}^n)$ and $V - v$ has a local maximum at x_0, then $\lambda V - \min_{a \in A} \{ f(x, a) \cdot Dv + l(x, a) \} \leq 0$ at x_0, and conversely for the minimum case. We specified in order to avoid any possible confusion due to the reversed inequality appearing in the finite-time case (because of the terminal condition)).

(8 Punkte)

In the following exercises we try to solve some optimal control situation with the help of the Pontryagin Maximum Principle (abbr. PMP). It can be directly applied when the considered cost is only at the end, but an extension to the general case is possible (here not discussed).

Theorem (Pontryagin Maximum Principle)**.** Consider the control system:

$$\dot{x} = f(x, \alpha), \qquad \alpha(t) \in A, \qquad t \in [0, T], \qquad x(0) = x_0$$

Let $t \mapsto \alpha^*(t)$ be an optimal control and $t \mapsto x^*(t) = x(t, \alpha^*)$ be the corresponding optimal trajectory for the maximization problem:

$$\max_{\alpha \in \mathcal{A}} \psi(x(T, \alpha))$$

Define the (row) vector $t \mapsto p(t)$ as the solution to the linear adjoint system:

$$\dot{p} = -p(t)B(t), \qquad B(t) \doteq D_x f(x^*(t), \alpha^*(t))$$

with terminal condition:

$$p(T) = \nabla\psi(x^*(T))$$

Then, for almost every $\tau \in [0, T]$ the following maximality condition holds:

$$p(\tau) \cdot f(x^*(\tau), \alpha^*(\tau)) = \max_{a \in A}\{p(\tau) \cdot f(x^*(\tau), a)\}$$

This result provides a necessary (but generally *not* sufficient) condition on an optimal strategy α^*. It can be actively used for building solutions by hands, by following this algorithm.
First, forget the time dependence $t \mapsto \alpha^*(t)$. Instead, find α^* as a function of x and p by exploiting the equality:

$$\alpha^*(x, p) = \underset{a \in A}{\operatorname{argmax}}\{p \cdot f(x, a)\}$$

Once you find $(x, p) \mapsto \alpha^*(x, p)$, plug this solution into the system:

$$\begin{cases} \dot{x} = f(x, \alpha^*(x, p)) \\ \dot{p} = -p \cdot D_x f(x, \alpha^*(x, p)) \end{cases}$$

with conditions $x(0) = x_0$, $p(T) = \nabla\psi(x(T))$.
It becomes a system with only variables t, x, p, that once solved gives us two functions $t \mapsto x(t)$ and $t \mapsto p(t)$. Finally, use them to construct $t \mapsto \alpha^*(t)$ via the composition $t \mapsto \alpha^*(x(t), p(t))$.

Exercise 2. (Linear Pendulum)

Let $q(t)$ be the position of a linearized pendulum with unit mass, controlled by an external force with magnitude $\alpha(t) \in [-1, 1]$. Then $q(\cdot)$ satisfies the second order ODE:

$$\ddot{q}(t) + q(t) = \alpha(t) \qquad q(0) = \dot{q}(0) = 0 \qquad \alpha(t) \in [-1, 1]$$

Problem: find an optimal strategy α^* that maximizes the terminal displacement $q(T)$.
(Hint: introduce variables $x_1 = q, x_2 = \dot{q}$...we seek $\max x_1(T, \alpha)$...the adjoint system in p can be solved without involving x or α...you can also avoid to solve $t \mapsto x(t)$...because maybe the max in the Pontryagin equation depends only on p_2...)

(4 Punkte)

Exercise 3. (The principle gives a condition necessary, but not sufficient)

Consider the following system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x_1} = \alpha \\ \dot{x_2} = x_1^2 \end{cases}$$

With initial conditions $x_1(0) = x_2(0) = 0$ and $\alpha(t) \in [-1, 1]$.

1 Find an optimal strategy to maximize $x_2(T)$.

2 Consider the control $\alpha(t) = 0$ for each time. Does it respect the Pontryagin Maximum Principle? Is it an optimal control?

(4 Punkte)

Exercise 4. (Compatibility between two definitions of consistency)

Check that if a scheme in differenced form is *consistent* w.r.t definition (26), then it is w.r.t. definition (19).

(4 Punkte)