V4E2 - Numerical Simulation

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Exercise sheet 7.

To be handed in on **Tuesday, 12.06.2017**.

In this sheet we prove more properties of the First-Order-Upwind and the Lax-Friedrichs schemes studied in class for the advection equation (CAE, VA).
As explained in the lecture after the Upwind scheme, a way to increase the consistency rate is to use a larger stencil of points. E first naive way would be given by the Forward Euler approximation, here written for the constant case:

$$v_j^{k+1} = v_j^k - \frac{c\Delta t}{2\Delta x}[v_{j+1}^k - v_{j-1}^k]$$

Exercise 1. (Euler scheme is unstable)

Prove that the previous Euler scheme is unstable.

(4 Punkte)

A solution to this inconvenience is suggested by the Lax-Friedrichs scheme (LF):

$$v_j^{k+1} = \frac{v_{j-1}^k + v_{j+1}^k}{2} - f(x_j, t_k)\frac{\Delta t}{2\Delta x}[v_{j+1}^k - v_{j-1}^k] + \Delta t g(x_j, t_k)$$

Exercise 2. (Lax-Friedrichs: consistency and CLF condition)

Prove that the previous LF scheme:

1 is consistent, by showing the bound for the local consistency error:

$$||L(\Delta; t, U(t))|| \leq C(\Delta t + \frac{\Delta x^2}{\Delta t})$$

2 satisfies the CFL condition when $\frac{||f||_\infty \Delta t}{\Delta x} \leq 1$

Hint: recall that for a smooth solution, it holds the approximation:

$$\frac{u(x_{j-1}, t_n) + u(x_{j+1}, t_n)}{2} = u(x_j, t_n) + O(\Delta x^2)$$

(6 Punkte)

Now we come back to the upwind scheme, checking better the bahavior of the numerical solution. First, recall the linear equation in one space dimension with constant coefficient $c > 0$

$$u_t(x,t) + cu_x(x,t) = 0, \quad u(x,0) = u_0(x). \tag{1}$$

The exact solution of the discretized equations (finite differences) satisfies a PDE which is generally different from the one to be solved:

Original equation		Modified equation solved by $u^{n+1} = S(\Delta, u^n)$
$\frac{\partial u}{\partial t} + \mathcal{L}u = 0$	$\sim$	$\frac{\partial u}{\partial t} + \mathcal{L}u = \sum_{p=1}^{\infty} \alpha_{2p}\frac{\partial^{2p} u}{\partial x^{2p}} + \sum_{p=1}^{\infty} \alpha_{2p+1}\frac{\partial^{2p+1} u}{\partial x^{2p+1}}$

Exercise 3. (Numerical solution for the Upwind scheme)

Prove that the numerical solution of (1) by the upwind scheme:

$$\frac{u_j^{i+1} - u_j^i}{\Delta t} + c\frac{u_j^i - u_{j-1}^i}{\Delta x} = 0$$

corresponds to a solution of:

$$u_t + cu_x = \frac{c\Delta x}{2}(1-\lambda)\frac{\partial^2 u}{\partial x^2} + \frac{c(\Delta x)^2}{6}(3\lambda - 2\lambda^2 - 1)\frac{\partial^3 u}{\partial x^3} + \dots$$

where $\lambda = \frac{c\Delta t}{\Delta x}$ and $\dots$ contains derivatives of order > 3. Hints:

- Expand all nodal values in the difference scheme in a double Taylor series about a single point (x_i, t_j) of the space-time mesh to obtain a PDE;
- Express high-order time derivatives as well as mixed derivatives in terms of space derivatives using this PDE to transform it into the desired form.

(5 Punkte)