V4E2 - Numerical Simulation

Sommersemester 2018
Prof. Dr. J. Garcke
Teaching assistant: Biagio Paparella
Tutor: Marko Rajković
(marko.rajkovic@uni-bonn.de)

Exercise sheet 8.

To be handed in on **Tuesday, 19.06.2018**.

This exercise sheet is devoted to the Upwind and LF schemes for Hamilton-Jacobi equations.

Exercise 1. (Monotonicity for the HJ-Upwind scheme)

In the lecture it has been introduced the upwind scheme $S^{up}(\Delta, V)$ for the convex Hamilton-Jacobi equation. Investigate monotonicity of this by deriving a condition on Δt and Δx to provide $\frac{\partial}{\partial v_i} S_j^{up}(\Delta, V) \geq 0$. In this scheme, does the CFL condition imply stability?
(Hint: the case $i \neq j$ is fine, but for $i = j$ you should obtain a condition similar to the one for the LF case...)

(5 Punkte)

Exercise 2. (Numerical implementation)

Consider the following one dimensional equation:

$$\begin{cases} u_t(x,t) + \frac{1}{2}|u_x(x,t)|^2 = 0, (x,t) \in (0,1) \times (0,T) \\ u(x,0) = u_0(x) \end{cases}$$

with $T = 0.05$ and two possible initial conditions with bounded support:

$$u_0(x) = \max(1 - 16(x-0.5)^2, 0)$$

and

$$u_0(x) = -\max(1 - 16(x-0.5)^2, 0)$$

They differ from a simple change of sign, but note that if the first if Lipschitz continuous, the second is also semiconcave. The exact solution $u(x,t)$ corresponding to the first initial condition is:

$$\begin{cases} \frac{(|x-\frac{1}{2}|-\frac{1}{4})^2}{2t}, & \frac{1}{4} \leq x \leq \frac{3}{4} \\ 0 & else \end{cases}$$

while the one corresponding to the second is:

$$u(x,t) = \min\left(\frac{|x-\frac{1}{2}|^2}{2t+\frac{1}{16}} - 1, 0\right)$$

Implement the Upwind and the LF scheme for solving this HJ equation with respect to both proposed initial conditions. For every initial condition, print the plot of the exact solution evolving in time, the plot of the Upwind approximation evolving in time and the one of the LF solution evolving in time (so, a total of 6 figures; you can use *pyplot* from the *matplotlib* library).
We suggest using a $\Delta t = 0.01$, and $\Delta x = 20\Delta t$. Fix a positive time of your choice (e.g. $t = 0.05$) and compute the errors in the infinite-norm (e.g. by using *norm* from *numpy.linalg* with parameter *np.inf*).
Does the choice of the initial condition seem to influence these results? Intuitively speaking, for which initial condition do you expect to find a higher converge results for $\Delta \to 0$? Why?

(13 Punkte)