V4E2 - Numerical Simulation

Sommersemester 2018
Prof. Dr. J. Garcke
Teaching assistant: Biagio Paparella
Tutor: Marko Rajković
(marko.rajkovic@uni-bonn.de)

UNIVERSITÄT BONN

Exercise sheet 9.

To be handed in on **Tuesday, 26.06.2018**.

The first part of this sheet is devoted to the weak notion of boundary condition, and we recap here the fundamental definitions. Then we move to numerical application with the the SL scheme for general advection equations. Finally, we introduce the Minimum Time Problem and show how it satisfies a Dynamic Programming Principle. At the very end, there is a *bonus* exercise concerning the Upwind scheme for 2-dimensional Hamilton-Jacobi equations.

Weak boundary conditions

Suppose that $u \in BUC(\Omega)$ is a viscosity solution for an HJ-equation in our usual setting. Suppose that $u \in BUC(\bar{\Omega})$, and let B be a suitable *boundary operator* defined accordingly the specific case. Then, u is said to solve the HJ equation with weak boundary condition B iff:

- *subsolution*: for any test function $\phi \in C^1(\bar{\Omega})$ and $x \in \partial\Omega$ local maximum for $u - \phi$:
$$\min\{H(x, u(x), D\phi(x)), B(x, u(x), D\phi(x))\} \leq 0$$
- *supersolution*: for any test function $\phi \in C^1(\bar{\Omega})$ and $x \in \partial\Omega$ local minimum for $u - \phi$:
$$\max\{H(x, u(x), D\phi(x)), B(x, u(x), D\phi(x))\} \geq 0$$

For example,$B(x, u(x), Du(x)) = u(x) - b(x)$ defines the weak Dirichlet boundary condition (i.e. the weak version of the classical request $u(x) = b(x)$ on $\partial\Omega$, the motivation has been explained in class via the vanishing viscosity example).

Exercise 1. (An example with weak Dirichlet condition)

Show that $u(x) = x$ is a viscosity solution of:

$$u'(x) = 1, u(0) = u(1) = 0$$

on the interval $\Omega = (0, 1)$

(4 Punkte)

Exercise 2. (A jumping back wave)

In this exercise we are going to implement the SL scheme for the coefficient-variable advection equation (VA) stated in class. Consider space and time intervals equal to $[0, 1]$ and

$$f(x, t) = sin(2\pi t)$$
$$g(x, t) = 0$$
$$u_0(x) = \max(1 - 16(x - 0.25)^2, 0)$$

Choose a $\Delta x = \Delta t = 0.01$. Plot the numerical solution evolving in multiple times (e.g. $t = 0, 0.25, 0.50, 0.75, 1.0$).

The analytic solution to the problem is given by

$$u(x, t) = u_0(-\frac{1}{2\pi} + x + \frac{\cos(2\pi t)}{2\pi})$$

Use this information for checking your solution and doing an error analysis.

(8 Punkte)

The minimum time problem

Description of the problem

We study problems with initial state $x \in \mathcal{T}^c := \mathbb{R}^N \setminus \mathcal{T}$, whose dynamics

$$\begin{cases} y'(t) = f(y(t), \alpha(t)),\ t > 0, \\ y(0) = x, \end{cases}$$

is stopped and the payoff computed when the system reaches the closed set $\mathcal{T}$, where int $\mathcal{T} \neq \emptyset$, $\partial\mathcal{T}$ is sufficiently regular. We are interested in the minimal time function

$$T(x) := \inf_{\alpha \in \mathcal{A}} t_x(\alpha)$$

where the first time of arrival is defined by

$$t_x(\alpha) := \begin{cases} +\infty & \text{if } \{t : y_x(t,\alpha) \in \mathcal{T}\} = \emptyset, \\ \inf\{t : y_x(t,\alpha) \in \mathcal{T}\} & \text{otherwise.} \end{cases}$$

Additionally we define the reachable set as

$$\mathcal{R} := \{x \in \mathbb{R}^N : T(x) < +\infty\},$$

which describes the set of initial states from which it is possible to reach the target $\mathcal{T}$.

Further prerequisites

Let $A \subset \mathbb{R}^M$ compact.

(A_0)

$$f : \mathbb{R}^N \times A \to \mathbb{R}^N \text{ is continuous,}$$

(A_3)

$$(f(x,a) - f(y,a)) \cdot (x - y) \leq L|x - y|^2 \text{ for all } x, y \in \mathbb{R}^N, a \in A.$$

Exercise 3. (The minimum time problem satisfies a Dynamic Programming Principle)

Assuming (A_0), (A_3), prove that for all $x \in \mathcal{R}$,

$$T(x) = \inf_{\alpha \in \mathcal{A}} \{\min\{t, t_x(\alpha)\} + \chi_{\{t \leq t_x(\alpha)\}} T(y_x(t,\alpha))\},$$

for all $t \geq 0$ and

$$T(x) = \inf_{\alpha \in \mathcal{A}} \{t + T(y_x(t,\alpha))\},$$

for all $t \in [0, T(x)]$.
PS: in the next sheet, there will be a second part concerning this exercise.

(6 Punkte)

Exercise 4. (*Bonus exercise*: HJ Upwind scheme in 2D)

We consider the 2-dimensional HJ equation:

$$u_t + H(u_{x_1}, u_{x_2}) = 0, \quad \mathbb{R}^2 \times [0,T] \tag{1}$$

Generate the upwind scheme for this setting. Prove (or give a reasonable sketch of) its monotonicity and consistency conditions.

(6* Punkte)