Scientific Computing II

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Sheet 1

Submission on **Thursday, 26.4.18**.

Exercise 1. (bilinear elements)

Consider the unit square $\Omega = [0,1]^2 \subset \mathbb{R}^2$. We call a continuous function $f\colon \Omega \to \mathbb{R}$ affine bilinear if $f(\cdot, y)$ is affine linear for all $y \in \Omega$ and $f(x,\cdot)$ is affine linear for all $x \in \Omega$.

a) Let $Q(\Omega)$ be the space of affine bilinear functions on Ω. Show that $Q(\Omega)$ has dimension 4, and find a basis which is nodal with respect to the corners of Ω.

Let $n \in \mathbb{N}$. We define $a_i = i/n$ for $i = 1, \dots, n$ and decompose Ω into a union of squares

$$\Omega_{ij} = \{(x,y)^\top \in \Omega \mid a_{i-1} \le x \le a_i,\, a_{j-1} \le y \le a_j\} \subset \Omega$$

for $i, j = 1, \dots, n$.

b) Find the dimension of

$$V = \{f \in \mathcal{C}(\Omega) \mid f|_{\Omega_{ij}} \in Q(\Omega_{ij}) \text{ for } i,j = 1,\dots,n\}$$

and determine whether a nodal basis with respect to the gridpoints $(a_i, a_j)^\top$, $i, j = 0, \dots, n$ exists.

(4 points)

Exercise 2. (stiffness- and mass matrix entries)

For given $n, i, j \in \mathbb{N}$, consider the space $Q(\Omega_{ij})$ given as in Exercise 1b). Let $\phi_1, \dots, \phi_4$ be the nodal basis of $Q(\Omega_{ij})$. Compute $\int_{\Omega_{ij}} \phi_k(x)\phi_l(x)\,\mathrm{d}x$ and $\int_{\Omega_{ij}} \nabla\phi_k(x)\nabla\phi_l(x)\,\mathrm{d}x$ for $k, l = 1 \dots, 4$.

(4 points)

Exercise 3. (Galerkin approximation)

Let $\Omega = [0,1]^2$ and consider the problem:
Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) := \int_\Omega \nabla u(x)\nabla v(x)\,\mathrm{d}x + \int_\Omega u(x)v(x)\,\mathrm{d}x = \int_\Omega v(x)\,\mathrm{d}x =: F(v)$$

for all $v \in H_0^1(\Omega)$.

a) Let $n \in \mathbb{N}$ and V be the space defined in Exercise 1b). Define $V_0 = \{v \in V \mid v = 0 \text{ on } \partial\Omega\}$ and consider the problem: Find $u \in V_0$ such that $a(u,v) = F(v)$ for all $v \in V_0$. Reformulate this problem as a linear system of equations using the Galerkin Ansatz.

b) Solve this linear system explicitely for $n = 3$.

(4 points)

Exercise 4. (oszillating coefficient)

Let $I = [0, 1]$ and consider the ODE

$$-[a_\epsilon(x)u'(x)]' = 1 \text{ in } I\,,$$
$$u(0) = u(1) = 0\,.$$

For $a_\epsilon(x) = (2 + \sin(x/\epsilon))^{-1}$, find a solution to the stated ODE and compute the corresponding effective coefficient

$$\left(\int_0^1 a_\epsilon(x)^{-1}\,\mathrm{d}x\right)^{-1}\,.$$

(4 points)