Scientific Computing II

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Sheet 2

Submission on **Thursday, 3.5.18.**

Let $\Omega \subset \mathbb{R}^n$ be an open domain and $Y = (0,1)^n$. Let $f \in L^2(\Omega)$ and $A \in \mathcal{A}_\sharp(\alpha, \beta, \Omega, Y)$, where $A(x,y) = A(y)$. We consider the problem:
Find $u^\epsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega A\left(\frac{x}{\epsilon}\right) \nabla u^\epsilon(x) \cdot v(x)\, \mathrm{d}x = \int_\Omega f(x) v(x)\, \mathrm{d}x \qquad (1)$$

holds for all $v \in H_0^1(\Omega)$.

Exercise 1. (asymptotic expansion I)

We assume that there exist smooth, Y-periodic functions $u_i(x,y)$, $i \in \mathbb{N}$ such that

$$u^\epsilon(x) = \sum_{i \in \mathbb{N}} \epsilon^i u_i\left(x, \frac{x}{\epsilon}\right).$$

Denote with div_y, ∇_y the corresponding differential operators with respect to the y-variable, as well as $\bar{y} = \frac{x}{\epsilon}$.

a) Calculate $\nabla u^\epsilon(x)$ in terms of the $(u_i)'s$, ordered by powers of ϵ.

b) Plug your result into equation (1) to obtain

$$\begin{aligned}
&\epsilon^{-2} && [\mathrm{div}_y(A(\bar{y})\nabla_y u_0(x,\bar{y}))] \\
&+\epsilon^{-1} && [\mathrm{div}_x(A(\bar{y})\nabla_y u_0(x,\bar{y})) + \mathrm{div}_y(A(\bar{y})(\nabla_x u_0(x,\bar{y}) + \nabla_y u_1(x,\bar{y})))] \\
&+ && [\mathrm{div}_y(A(\bar{y})(\nabla_x u_1(x,\bar{y}) + \nabla_y u_2(x,\bar{y}))) + \mathrm{div}_x(A(\bar{y})(\nabla_x u_0(x,\bar{y}) + \nabla_y u_1(x,\bar{y})))] \\
&+\mathcal{O}(\epsilon) && = -f(x).
\end{aligned}$$

(4 points)

Exercise 2. (asymptotic expansion II)

Assume that the equation from Exercise 1b) holds for general $y \in Y$ and equate coefficients to obtain the following differential equations:

- $$-\mathrm{div}_y(A(y)\nabla_y u_0(x,y)) = 0$$
  with $u_0(x,y)$ is Y-periodic. Conclude that $u_0(x,y) = u_0(x)$ is not depending on $y \in Y$.

- $$\mathrm{div}_y(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y))) = 0$$
  with $u_1(x,y)$ is Y-periodic. Conclude that one can write
  $$u_1(x,y) = u_1(x) + \nabla_x u_0(x) \cdot w(y)$$
  with $w \colon Y \to \mathbb{R}^n$ is Y-periodic, where $w_i(y)$ satisfies
  $$\mathrm{div}_y(A(y)(\nabla_y w_i(y) + e_i)) = 0$$
  for $i = 1, \dots, n$ (assume that such w_i exist).

- $$-f(x) = \mathrm{div}_y(A(y)(\nabla_x u_1(x,y) + \nabla_y u_2(x,y))) + \mathrm{div}_x(A(y)(\nabla_x u_0(x) + \nabla_y u_1(x,y)))$$

  Integrate this equation with respect to Y and show that the first summand on the right hand side vanishes due to a periodicity argument. For the second term, plug in the representation for $u_1(x,y)$ and conclude that

  $$f(x) = -\,\mathrm{div}_x(A^0 \nabla_x u_0(x))$$

  with

  $$A^0_{ij} = \int_Y A(y)(e_j + \nabla_y w_j(y))\,\mathrm{d}y \cdot e_i\,.$$

(6 points)

Exercise 3. (periodic boundary problem)

Let $Y = (0,1)^n$, $f \in L^2(Y)$ and consider the problem:
Find $u \in \tilde{H}^1_\sharp(Y)$ such that

$$\int_Y \nabla u(y) \nabla v(y)\,\mathrm{d}y = \int_Y f(y) v(y)\,\mathrm{d}y$$

for all $v \in \tilde{H}^1_\sharp(Y)$.
Here,

$$\tilde{H}^1_\sharp(Y) = \{v \in H^1(Y) \mid v \text{ has periodic boundary conditions and zero mean}\}$$

equipped with the usual H^1-norm. Show that this problem has a unique solution u, which depends continuously on f. Show that if $u \in C^2(Y)$, it solves the PDE $-\Delta u = f$ in the strong sense.

(6 points)