Scientific Computing II

Summer term 2018
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Sheet 4

Submission on **Thursday, 17.5.18.**

Exercise 1. (heterogeneous multiscale method)

We consider an open, and bounded domain Ω with triangulation $T \in \mathcal{T}_H$ and continuous, piecewise linear finite elements $V_H(\Omega)$ with zero boundary, as well as $Y = (0,1)^n$ with triangulation $K \in \mathcal{T}_h$ and continuous, piecewise linear, periodic, zero-mean finite elements $W_h(Y)$. Moreover, let $A \in C^0(\overline{\Omega \times Y})^n$ be periodic in its second variable, uniformly elliptic and $A^\epsilon(x) = A(x, x/\epsilon)$. We use the piecewise constant approximation on inner cells

$$A_h^\epsilon(x)|_{x_T^\epsilon(K)} = A(x_T, x_T^\epsilon(y_K)/\epsilon)$$

for $T \in \mathcal{T}_H$ and $K \in \mathcal{T}_h$ with corresponding barycenters x_T, y_K.
$u_H \in V_H(\Omega)$ is called an HMM-approximation if it solves

$$(f, v_H)_{L^2(\Omega)} = \mathcal{A}_h(u_H, v_H) \quad \forall v_H \in V_H(\Omega)\,,$$

where

$$\mathcal{A}_h(u_H, v_H) = \sum_{T \in \mathcal{T}_H} |T| \fint_{Y_{T,\epsilon}} A_h^\epsilon(x) \nabla_x R_T(u_H)(x) \cdot \nabla_x v_H(x)\, \mathrm{d}x\,.$$

(Here, R_T is the local reconstruction operator for $\delta = \epsilon$.)
Show that u_H is the coarse part of the solution to the following two-scale problem:
Find $(u_H, u_h) \in V_H(\Omega) \times V_H(\Omega, W_h(Y))$ with

$$\int_\Omega \int_Y A_H(x,y)(\nabla_x u_H(x) + \nabla_y u_h(x,y)) \cdot (\nabla_x v_H(x) + \nabla_y v_h(x,y))\, \mathrm{d}y\, \mathrm{d}x = \int_\Omega f(x) v_H(x)\, \mathrm{d}x$$

for all $(v_H, v_h) \in V_H(\Omega) \times V_H(\Omega, W_h(Y))$, with $A_H(x,y)|_{T \times Y} = A(x_T, y)$.
Show that one additionally has

$$u_h(x,y)|_{T \times Y} = \frac{1}{\epsilon}(R_T(u_H) - u_H) \circ x_T^\epsilon(y - w_0)$$

with $w_0 = x_T/\epsilon + (1/2, \ldots, 1/2)^\top$.

(16 points)