# Scientific Computing II

Summer term 2018
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## Sheet 7

Submission on **Thursday, 28.6.18**.

**Exercise 1.** (Helmholtz equation)

Let $\Omega = [0, 1]$ and consider the Helmholtz equation

$$\begin{aligned} -u'' &= \lambda u \quad \text{in } \Omega \\ u(0) &= u(1) = 0 \end{aligned}$$

for $\lambda \in \mathbb{R}$.

a) Let $(u, \lambda)$ be a strong solution. Show that $u \in C^\infty(\Omega)$.

b) Let $(u, \lambda)$, $(v, \mu)$ be Eigenpairs with $\lambda \neq \mu$. Show that $(u, v)_{L^2(\Omega)} = 0$.

c) Compute all Eigenpairs $(u, \lambda)$.

(6 points)

**Exercise 2.** (Laplacian)

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and consider the Laplacian as a linear operator acting on $H_0^1(\Omega)$. Show that the Eigenfunctions $(\phi_i, \lambda_i)_{i=1,\dots,\infty}$ of $(-\Delta)$ on are an orthogonal basis of $H_0^1(\Omega)$. Show that the Eigenvalues are bounded from below by a constant $c > 0$.

(4 points)

**Exercise 3.** (ONB expansion)

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain and $T > 0 \in \mathbb{R}$. Let $L\colon H_0^1(\Omega) \longrightarrow L^2(\Omega)$ be a linear, continuous, elliptic operator. Let $(\phi_i, \lambda_i)_{i=1,\dots,\infty}$ be an orthonormal basis of $L^2(\Omega)$ of Eigenpairs to $L$. Consider the parabolic equation

$$\begin{aligned} \partial_t u + Lu &= f \quad \text{in } \Omega \times [0, T] \\ u(\cdot, 0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

with data $u_0 \in L^2(\Omega)$, $f \in L^2(\Omega \times [0, T])$. Show that a solution can be written as

$$u(x, t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} (u_0, \phi_i)_{L^2(\Omega)} \phi_i(x) + \sum_{i=1}^{\infty} \int_0^t e^{-\lambda_i (t-s)} (f(\cdot, s), \phi_i)_{L^2(\Omega)} \, \mathrm{d}s \, \phi_i(x) \, .$$

(6 points)