Scientific Computing II

Summer term 2018
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Sheet 8

Submission on **Thursday, 5.7.18**.

The goal of this exercise sheet is to resolve singular initial conditions for parabolic equations, using a geometrically refined time mesh. We consider the model problem from the lecture

$$\begin{aligned} u'(t) + Lu(t) &= g(t), \quad t \in I = (0,1) \\ u(0) &= u_0 \end{aligned}$$

with $L(X, X^*)$ an elliptic differential operator, $u_0 \in X$, and $g \in L^2(I, H)$ being analytic as a function in time. We discretize I with the partition $\mathcal{T}_{n,\sigma} = \{I_m\}_{m=1}^{n+1}$ with grading factor $\sigma \in (0,1)$ and $n+1$ time intervalls I_m given by the nodes $t_0 = 0$, $t_m = \sigma^{n-m+1}$. The time steps $\Delta t_m = t_m - t_{m-1}$ satisfy $\Delta t_m = \lambda t_{m-1}$, $\lambda = \frac{\sigma}{1-\sigma}$. Set $\gamma = \max\{1, \lambda\}$. Assign to every interval a polynomial approximation order r_m and assume regularity of the solution $u|_{I_m} \in H^{s_m}$ with $s_m = \alpha_m r_m$, $\alpha_m \in (0,1)$.

Exercise 1. (sharper estimate away from the initial conditions)

Fix an interval I_m, $m \geq 2$. Show that there exist constants $C, d > 0$ s.t.

$$\|u - \Pi_{I_m}^{r_m} u\|_{L^2(I_m, X)}^2 \leq C\sigma^{(n-m+2)} \left((\gamma d)^{2\alpha_m} \left[\frac{(1-\alpha_m)^{1-\alpha_m}}{(1+\alpha_m)^{1+\alpha_m}} \right] \right)^{r_m}$$

with C, d only depending on u_0, g.

(10 points)

Exercise 2. (exponential convergence)

Let u be the exact solution and $U \in \mathcal{V}^{\underline{r}}(\mathcal{T}_{n,\sigma}, X)$ the computed hp-DGFEM solution. Show that there exists $\mu > 0$ s.t. for $r_m = \lfloor \mu m \rfloor$, $m = 1, \ldots, n+1$, one obtains

$$\|u - U\|_{L^2(I,X)}^2 \leq C \exp(-bN^{\frac{1}{2}})$$

with constants C and b independent of $N = \dim \mathcal{V}^{\underline{r}}(\mathcal{T}_{n,\sigma}, X)$.
You can use the following estimate for the first interval without proof:

$$\|u - U\|_{L^2(I_1,X)}^2 \leq c\sigma^n ,$$

with $c > 0$ independent of n.

(10 points)