Sheet 9

Submission on **Thursday, 12.7.18.**

Programmieraufgabe 1. (moving least squares)

Let $\Omega = [0,1]^2$ and $f\colon \Omega \longrightarrow \mathbb{R}$ be given by

$$f(x) = \begin{cases} 1 & 1/2 \leq \|x\|_2 \leq 1\,, \\ 0 & \text{else}\,. \end{cases}$$

We want to approximate f with a smooth function using the moving least squares algorithm.

a) Write a routine that generates N uniformly distributed random samples $\{x_i\}_{i=1}^N \subset \Omega$ and stores them in a vector $x \in (\mathbb{R}^2)^N$.

To allow fast access to the random samples based on their location, we use a binning procedure:

b) Write a routine that takes $x \in (\mathbb{R}^2)^N$ and $M \in \mathbb{N}$ as input and produces a set of M^2 vectors $y_{ij} \in (\mathbb{R}^2)^{n_{ij}}$, $i,j = 1,\dots,M$ satisfying (thinking of x, y_{ij} as sets)

- $\sum_{i,j=1}^M n_{ij} = N$
- $\bigcup_{i,j=1}^M y_{ij} = x$
- $y_{ij} \subset \Omega_{ij} = [\frac{i-1}{M}, \frac{i}{M}] \times [\frac{j-1}{M}, \frac{j}{M}]$ for $i,j = 1,\dots,M$

As a weight function, we use

$$\theta(d) = \begin{cases} (1-dM)^4(4dM+1) & d \leq 1/M\,, \\ 0 & \text{else}\,. \end{cases}$$

Therefore the minimization of

$$\sum_{i=1}^N \theta(\|z - x_i)\|)|f(x_i) - p(x_i)|^2$$

over a given polynomial space can be performed on the 3x3 Bin-patch surrounding z.

c) Write a routine that takes $z \in \Omega$, $M \in \mathbb{N}$, $\{y_{ij}\}_{i,j=1}^M$ and $p \in \mathbb{N}$ and returns the MLS-approximation of $f(z)$ with order p (we take monomial basis functions $P_{qr}(z) = z_1^q z_2^r$, $q + r \leq p$). This includes the following steps:

- Find i,j such that $z \in \Omega_{ij}$, and assemble the points $Z = \bigcup_{k,l\in\{-1,0,1\}} y_{i+k,j+l} = \{z_a\}_{a=1}^m$
- assemble the Vandermonde matrix $V \in \mathbb{R}^{m\times(p+1)(p+2)/2}$, $V_{a,qr} = P_{qr}(z_a)$ for $a = 1,\dots,m$, $q + r \leq p$

- assemble the weight matrix $W \in \mathbb{R}^{m\times m}$, $W_{ab} = \delta_{ab}\theta(\|z - z_a\|)$ for $a, b = 1, \dots, m$
- Solve the linear system

  $$V^\top W F = V^\top W V b$$

  with $F \in \mathbb{R}^m$, $F_a = f(z_a)$ for $a = 1, \dots, m$. Do this with an iterative solver (e.g. CG-method or Jacobi method) and with the starting point $b^0 = (1/2, 0, \dots, 0)^\top$
- return the MLS approximation $f(z) \approx \sum_{q+r\leq p} b_{qr} P_{qr}(z)$

d) Test your implementation for $N \in \{10, 100, 1000, 10000\}$, $M = \lfloor N^{1/3} \rfloor$, $p = 3$ and plot your solution using an equidistant rectangular sampling grid of size 201×201.

(20 points)

The programming exercise should be handed in either before/after the exercise class on 12.7.18 (bring your own laptop!) or in the HRZ-CIP-Pool, after making an appointment at 'angelina.steffens@uni-bonn.de'. All group members need to attend the presentation of your solution. Closing date for the programming exercise is the 12.7.2018. You can choose the programming language yourself.