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Exercise Sheet 7.

Due date: **28.05.2019**.

Exercise 1. (A perturbation of A generates a weaker Hilbert scale) (5 Points)

Let A be a symmetric positive, and $B = (I + A)^{-1}A$. Prove that:

$$|||Bx|||_s \leq |||x|||_s \tag{1}$$

for all $s \in \mathbb{R}$.

Ps: the norm $|||\cdot|||_s$ is the one defined in class.

Exercise 2. (The stiffness matrix has always been there) (5 Points)

Let X, Y, Z be real vector spaces, $a(\cdot, \cdot) : X \times Y \to Z$ be a bilinear map, $\psi \in X^m$ and $\phi \in Y^n$. We define the associated matrix in $Z^{m \times n}$ as:

$$[a(\psi, \phi)]_{ij} = a(\psi_i, \phi_j) \tag{2}$$

calling it still a with a small abuse of notation.

Similarly, if $l : X \to Z$ is a linear map and $\psi \in X^{p \times q}$, define the matrix in $Z^{p \times q}$:

$$[l(\psi)]_{ij} = l(\psi_{ij}) \tag{3}$$

Fix two matrices $B \in \mathbb{R}^{s \times m}$, $C \in \mathbb{R}^{t \times n}$ and prove the followings:

i) if $\psi \in X^m$, $\phi \in Y^n$, then $a(B\psi, C\phi) = Ba(\psi, \phi)C^T$

ii) if $\psi \in X^{m \times 1}$, then $l(B\psi) = Bl(\psi)$

iii) if $\phi \in X^{1 \times n}$, then $l(\phi C^T) = l(\phi)C^T$

iv) if $\psi \in S_h^N$ is basis of the $N-$dim Galerkin approximation of H_0^1, then the solution of the variational problem can be written as $\bar{x} = a(\psi, \psi)^{-1} l(\psi)$

v) let's go in the setting of the multigrid algorithm, with ψ still to be the basis above with stiffness matrix A_h. Consider the basis for the level S_H done by $\psi_H = P^T \psi_h$. Show that the corresponding stiffness matrix actually coincides with the definition $A_H = P^T A_h P$ given in class.

Exercise 3. (Missing details in Theorem 2.25) (5 Points)

In the setting of theorem 2.25, prove by induction that:

$$\rho_1^2 + (\frac{6}{5}\rho_1)^4 (1 - \rho_1^2) \leq (\frac{6}{5}\rho_1)^2 \tag{4}$$

Exercise 4. (Missing details in Proposition 2.26) (5 Points)

In the setting of theorem 2.26, prove that:

$$\sum_i \lambda_i \mu_i^{2\nu} |c_i|^2 \leq [\sum_i \lambda_i \mu_i^{2\nu+1} |c_i|^2]^{\frac{2\nu}{2\nu+1}} [\sum_i \lambda_i |c_i|^2]^{\frac{1}{2\nu+1}} \tag{5}$$

Hint: Hoelder inequality