Scientific Computing II

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Exercise sheet 0.

Hand-in: -.

This exercise sheet is a warm up (without points for the exam admission). The exercises are intended to recall some concepts which you should have seen before during your studies and which will help you understand the contents of this course.

Real Hilbert spaces: basic notions

Definition 1 (Scalar product, pre-Hilbert space)**.** Let E be a vector space over $\mathbb{R}$. The map $\langle\cdot,\cdot\rangle : E \times E \to \mathbb{R}$ is called *scalar product* or *inner product* on E if

(S1) $x \mapsto \langle x, y\rangle$ is linear for all $y \in E$,

(S2) $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in E$.

(S3) $\langle x, x\rangle \geq 0$ for all $x \in E$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

The pair $(E, \langle\cdot,\cdot\rangle)$ or simply E is called a *pre-Hilbert space.*

Every pre-Hilbert space $(E, \langle\cdot,\cdot\rangle)$ is a normed space with norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Definition 2. A complete pre-Hilbert space is called a *Hilbert space.*

Definition 3. Let E be a pre-Hilbert space. A countable subset $S \subset E$ is called *orthonormal system* (ONS) if for all $x, y \in S$ we have

$$\langle x, y\rangle = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y. \end{cases}$$

An orthonormal system S is called *orthonormal basis* (ONB) if $\operatorname{span}(S)$ is dense in E.

Exercise 1. (Cauchy-Schwarz inequality)

Let E be a pre-Hilbert space over $\mathbb{R}$ with inner product $\langle\cdot,\cdot\rangle$. Show that for all $x, y \in E$

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle \langle y, y\rangle.$$

Make sure you understand the geometrical meaning of the above inequality.

(0 points)

Exercise 2. (Hilbert space of sequences)

Consider the sequence space

$$\ell_2(\mathbb{N}) = \left\{ (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\}.$$

Prove the following claims:

a) The map $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ defines an inner product on $\ell_2(\mathbb{N})$.

b) The space $\ell_2(\mathbb{N})$ with inner product $\langle\cdot,\cdot\rangle$ is a Hilbert space.

(0 points)

Exercise 3. (Bessel's inequality)

Let E be a pre-Hilbert space and $(e_i)_{i\in\mathbb{N}}$ be a ONS in E. Show that for all $x \in E$

$$\sum_{i=1}^{\infty} |\langle x, e_i\rangle|^2 \leq \|x\|^2.$$

Discuss the geometrical meaning of the inequality.

(0 points)

Exercise 4. (ONB properties)

Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and further, let S be a ONS in H. Prove that the following statements are equivalent:

1. S is a ONB.
2. For $x, y \in H$ we have $\langle x, y\rangle = \sum_{e\in S}\langle x, e\rangle\langle e, y\rangle$.
3. For all $x \in H$ we have $\|x\|^2 = \sum_{e\in S} |\langle x, e\rangle|^2$.
4. $S^{\perp} = \{x \in H : \langle x, e\rangle = 0 \text{ for all } e \in S\} = \{0\}$.
5. S is a maximal ONS, that is, there is no ONS S' such that $S \subsetneq S'$.
6. $\mathrm{span}(S)$ is dense in H.

(0 points)