Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn

universitätbonn

Computer lab Numerical Algorithms
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Prof. Dr. M. Rumpf – B. Geihe, B. Heeren

Problem sheet 5 **December 12th, 2012**

Problem 6 (Volume and perimeter length of phase fields)

The task of this exercise will be to compute the volume and perimeter length of a given object represented by a phase field v. In our lab code a phase field will simply be treated as a piece wise linear finite element function. To compute volume and perimeter length the following integrals need to be evaluated:

$$\mathcal{V}[v] = \int_D \tfrac{1}{4}(v+1)^2 \,\mathrm{d}x$$
$$\mathcal{L}^\epsilon[v] = \tfrac{1}{2}\int_D \epsilon\|\nabla v\|^2 + \tfrac{1}{\epsilon}\tfrac{9}{16}(v^2-1)^2 \,\mathrm{d}x$$

Using the standard finite element procedure one obtains for a discrete approximation V of the phase field:

$$\begin{aligned}
\mathcal{V}[V] &= \tfrac{1}{4}\int_D \left(\sum_i \bar{V}_i\psi_h^i\right)^2 + 2\left(\sum_i \bar{V}_i\psi_h^i\right) + \left(\sum_i \mathbb{1}_i\psi_h^i\right) \mathrm{d}x \\
&= \tfrac{1}{4}\left(\bar{V}^T M\bar{V} + 2\bar{V}^T M\mathbb{1} + \mathbb{1}^T M\mathbb{1}\right) \\
\mathcal{L}^\epsilon[V] &= \tfrac{1}{2}\int_D \epsilon\|\sum_i \bar{V}_i\nabla\psi_h^i\|^2 + \tfrac{1}{\epsilon}\tfrac{9}{16}\left((\sum_i \bar{V}_i\psi_h^i)^2 - 1\right)^2 \mathrm{d}x \\
&= \tfrac{\epsilon}{2}\bar{V}^T L\bar{V} + \tfrac{1}{2\epsilon}\tfrac{9}{16}\int_D \left((\sum_i \bar{V}_i\psi_h^i)^2 - 1\right)^2 \mathrm{d}x
\end{aligned}$$

That means that all terms except the last one can be computed by using mass and stiffness matrices. For the remaining term a new class NonlinearFEOperator will be provided. Similar to the linear case a separate class for the evaluation of the integrand has to be provided. It must implement the method addEval which evaluates the integrand at a given quadrature point, multiplies the results by the given scaling factor w and adds it to the return value b. See the classes provided for an example and further details.
Of course the concept of this nonlinear operator can also be used to evaluate the whole functional. This would save the cost for matrix assemblies but would make each evaluation more involved.

Tasks:

(i) Use the provided routines to create phase fields for a circle and a bar.

(ii) Compare your approximation to the known true values.

(iii) Try to decrease the error as much as possible by:

- Changing ε and the global mesh size h. What can be said about their relation?
- Doing local adaptive mesh refinements. For this purpose you need to implement an error indicator like on problem sheet 2. Think about where the crucial regions of the phase fields are and how they could be detected by an error indicator.

Problem 7 (Deformation of elastic objects)

In this task an elastic object represented by a phase field will be exposed to an external loading. The goal is to approximate its response behavior based on the partial differential equations of linearized elasticity. On problem sheet 6 the resulting linear system of equations was derived:

$$\begin{pmatrix} E_{11} & E_{21} & \cdots \\ E_{21} & E_{22} & \\ \vdots & & \ddots \end{pmatrix} \begin{pmatrix} U^1 \\ U^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} G^1 \\ G^2 \\ \vdots \end{pmatrix}, \quad \begin{aligned} E_{kl} &= \int_D C\,\varepsilon[\psi_i^k] : \varepsilon[\psi_j^l]\,\mathrm{d}x \\ &= \int_D \left(\lambda\,\mathrm{tr}\,\varepsilon[\psi_i^k]\mathbb{1} + 2\mu\varepsilon[\psi_i^k]\right) : \varepsilon[\psi_j^l]\,\mathrm{d}x \end{aligned}$$

In this task we however would like to prescribe a surface loading g on a part $\Gamma_N \subset \partial\mathcal{O}$ of the object whereas we will have homogeneous Dirichlet boundary data on $\Gamma_D \subset \partial\mathcal{O}$ and homogeneous Neumann boundary data $g \equiv 0$ on the remaining part $\Gamma_0 = \partial\mathcal{O} \setminus (\Gamma_D \cup \Gamma_N)$. For the right hand side a boundary integral $G^k := \left(\int_{\Gamma_N} g_k \cdot \varphi_i^h\right)_i$ has to be evaluated on Γ_N. To avoid setting up a new generic integral operator for 1D computations this may be done manually by using the BoundaryIterator and computing the integrals on the segments explicitly. For the Dirichlet data the already known DirichletBoundaryMask will be used. The remaining part Γ_0 does not require any further treatment.
As the library will be updated to support vector valued problems there will be a new dimension template parameter Dim for the LinearFEOperator and additional arguments for the bilinear forms.

Tasks:

(i) Complete the bilinear form PhasefieldElastStiffBilf needed to set up the stiffness matrix for the linearized elastic problem. It needs to receive an external phase field (with values in $[-1,1]$) upon construction which will be used to interpolate between the usual Lamé-Navier tensor A and a weak material B:

$$\tilde{v} = \tfrac{1}{2}v + \tfrac{1}{2}, \quad C := \tilde{v}A + (1-\tilde{v})B$$

(ii) Test your implementation using the main program provided.