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Computer lab Numerical Algorithms
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Problem sheet 7 **January 23rd, 2012**

Problem 10 (Phase field shape optimization)

In this exercise we will finally treat a phase field shape optimization model. As in exercise 7 the phase field function v determines regions within the working domain in which a strong material is used whereas there is weak material in the other regions. To rule out trivial solutions the overall volume of the strong material is penalized. Additionally we will use a regularization for the perimeter length of these regions. Hence we consider the following energy for minimization:

$$\begin{aligned}
J[v,u] &= J_{\text{phsy}}[v,u] + \nu\mathcal{V}[v] + \eta\mathcal{L}^\epsilon[v] \\
J_{\text{phsy}}[v,u] &= \tfrac{1}{2}\int_D \lambda\,(\operatorname{tr}\varepsilon[u])^2 + 2\mu\,\varepsilon[u] : \varepsilon[u]\,\mathrm{d}x \\
\lambda &= \chi(v)\lambda^1 + (1-\chi(v))\lambda^2 \quad \mu = \chi(v)\mu^1 + (1-\chi(v))\mu^2 \\
\mathcal{V}[v] &= \int_D \chi(v)\,\mathrm{d}x \\
\mathcal{L}^\epsilon[v] &= \tfrac{1}{2}\int_D \epsilon\|\nabla v\|^2 + \tfrac{1}{\epsilon}\tfrac{9}{16}(v^2-1)^2\,\mathrm{d}x \\
\chi(v) &= \tfrac{1}{4}(v+1)^2
\end{aligned}$$

The displacement u has to fulfill the partial differential equations of linearized elasticity, i.e. it is the unique minimizer

$$u = \operatorname*{arg\,min}_{\tilde{u}\in \overset{0}{H}{}^{1,2}_{\Gamma_D}} E[v,\tilde{u}] = \operatorname*{arg\,min}_{\tilde{u}\in \overset{0}{H}{}^{1,2}_{\Gamma_D}} \tfrac{1}{2}\int_D \lambda\,(\operatorname{tr}\varepsilon[\tilde{u}])^2 + 2\mu\,\varepsilon[\tilde{u}] : \varepsilon[\tilde{u}]\,\mathrm{d}x - \int_{\Gamma_N} g\cdot\tilde{u}\,\mathrm{d}a\,.$$

It will be computed by solving the weak formulation with piece wise linear finite elements as in exercise 7.

To perform the optimization we will use the gradient descent algorithm that was implemented in exercise 8. The general theory tells us that the gradient is given by

$$J_{,v}[v,u](s) = J_{\text{phsy},v}[v,u](s) + \nu\mathcal{V}[v]_{,v}(s) + \eta\mathcal{L}^\epsilon[v]_{,v}(s) - E_{,vu}[v,u](p)(s)$$

where p solves the dual problem

$$E_{,uu}[v,u](p)(\varphi) = J_{,u}[v,u](\varphi) = J_{\text{phsy},u}[v,u](\varphi) \quad \forall\,\varphi\in \overset{0}{H}{}^{1,2}_{\Gamma_D}\,.$$

Due to the special choice of J_{phsy} in this exercise we however see that $E_{,uu}[v,u](p)(\varphi)$ and $J_{\text{phys},u}[v,u](\varphi)$ are actually exactly the same expression except for the variables u,p. We therefore must have $p = u$ and the dual solution does not need to be computed. Moreover we have $E_{,u}[v,u](p) = E_{,u}[v,u](u) = 2\,J_{\text{phsy}}[v,u]$. Thus we finally obtain for the gradient:

$$J_{,v}[v,u](s) = -\tfrac{1}{2} J_{\text{phsy},v}[v,u](s) + \nu \mathcal{V}[v]_{,v}(s) + \eta \mathcal{L}^{\epsilon}[v]_{,v}(s)$$

For implementation we propose the following framework: There will be two operators PhasefieldElasticEnergy and PhasefieldElasticEnergyGrad which will be passed to the GradientDescent class to compute the energy $J[v,u]$ and the gradient $(J_{,v}[v,u](\varphi_i))_i$. Both operators need to access the primal solution u for the currently given phase field they are applied to. To centralize the computation of u all relevant code of exercise 7 has been moved to a separate class ElasticProblem which will be passed to both operators. Its method updatePhasefield should be used to hand over the current phase field the operators are applied to. The ElasticProblem class will then check if there was a modification to the phase field since the last computation of u and recompute it if need be. The solution can then be accessed through the method getSolution.

Task:
Evaluate the discrete energy $J[V,U]$ and gradient $(J_{,V}[V,U](\varphi_i))_i$ in the corresponding applyAdd methods of PhasefieldElasticEnergy and PhasefieldElasticEnergyGrad. For the discretization of $\mathcal{V}[v]$ and $\mathcal{L}^{\epsilon}[v]$ refer to exercise 6 and for J_{phsy} to exercise 7. However note that the characteristic function has changed to match the notation of the lecture.