Institut für Numerische Simulation
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Tutorial Numerical Algorithms
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Problem sheet 2 — October 23rd, 2012

Problem 1

Consider a d-dimensional simplex $T \subset \mathbb{R}^d$, i.e.

$$T = T[x_0, \dots, x_d] = \Big\{ \sum_{i=0}^{d} \lambda_i x_i \in \mathbb{R}^d \, : \, \lambda_i \geq 0, \sum_i \lambda_i = 1 \Big\} \,,$$

with $h(T) \leq h$ and $\rho(T) \geq \kappa h$, and a $(d-1)$-dimensional subsimplex $F \subset \partial T$, i.e.

$$F = F_l = \Big\{ \sum_{i=0}^{d} \lambda_i x_i \in T \, : \lambda_l = 0 \Big\}$$

for some $l \in \{0, \dots, d\}$. Show that for $u \in H^{1,2}(T)$:

$$\|u\|_{L^2(F)} \leq C(\kappa)(h^{-\frac{1}{2}} \|u\|_{L^2(T)} + h^{\frac{1}{2}} \|\nabla u\|_{L^2(T)}).$$

Remark: Perform a transformation onto the reference domain (cf. Thm 1.4, *scaling argument*) and use that for bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ there is a bounded linear operator $B : H^{1,p}(\Omega) \to L^p(\partial\Omega)$ with $Bu = u$ on $\partial\Omega$ for all $u \in H^{1,p}(\Omega) \cap C^0(\bar{\Omega})$. B is called the *trace operator*.

Problem 2

Under the assumptions of problem 1, show that for $u \in H^{\mathrm{k}+1,2}(T)$, $k \geq 1$:

$$\|u - \mathcal{I}_h u\|_{L^2(F)} \leq C(\kappa) h^{\mathrm{k}+\frac{1}{2}} \|u\|_{H^{\mathrm{k}+1,2}(T)}.$$

Here $\mathcal{I}_h$ denotes the interpolation operator on $P_{\mathrm{k}}(T)$.

Problem 3

Let H be a Hilbert space with subspaces $V_h \subset V \subset H$. Consider coercive, bounded, bilinear forms $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ and bounded linear forms l and l_h on V and V_h, respectively. Furthermore, u and u_h denote weak solutions of

$$a(u,v) = l(v) \quad \text{for all } v \in V$$

and

$$a_h(u_h, v_h) = l_h(v_h) \quad \text{for all } v_h \in V_h\,.$$

Show that

$$\|u-u_h\|_H \le \sup_{g\in H} \frac{1}{\|g\|_H} \quad \inf_{\varphi_h\in V_h}\Big(\quad C\|u-u_h\|_V\|\varphi_g-\varphi_h\|_V + \\ |a(u_h,\varphi_h)-a_h(u_h,\varphi_h)| + |l(\varphi_h)-l_h(\varphi_h)|\,\Big)\,,$$

where $\varphi_g \in V$ for $g \in H$ denotes the (weak) solution of the dual problem

$$a(v,\varphi_g) = (g,v)_H \quad \text{for all } v \in V\,.$$

Problem 4

The local interpolation estimate of Theorem 1.4 reads

$$\|u-\mathcal{I}_h u\|_{m,q,T} \le Ch(T)^{k+1-m+d(\frac{1}{q}-\frac{1}{p})}|u|_{k+1,p,T} \quad \text{for all } u \in H^{k+1,p}(T)$$

for appropriate assumptions. For an admissible triangulation $\mathcal{T}_h$ of some domain $D \subset \mathbb{R}^d$ the global interpolation estimate of Corollar 1.5 states

$$\|u-\mathcal{I}_h u\|_{m,p,D} \le C\,h^{k+1-m}|u|_{k+1,p,D} \quad \text{for all } u \in H^{k+1,p}(D)\,.$$

Is there a better global estimate (in terms of powers of h) for $p > q$?