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Tutorial Numerical Algorithms

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Problem sheet 3 **October 30th, 2012**

Problem 1 (Red-Green refinement)

Red-Green refinement is based on the bisection of all edges in a marked triangle in one step, e.g. replacing this triangle by four (red refinement). To avoid "hanging nodes" in an adaptive grid one introduces a temporary so-called green refinement. This is realized by introducing additional edges in triangles that were affected by a green refinement of one or two neighboring triangles. However, this temporary green refinement should be replaced by a red refinement if the neighboring red-refined triangle is refined again in the next step.

Think about possible refinement patterns and write an algorithm to refine a grid which is already Red-Green refined.

Problem 2 (Local projection on polynomials)

Why is the local L^2–projection defined on the local FE–space instead of an appropriate space of polynomials? Consider the $1D$-example with reference domain $\hat{\omega} = [-1, 1]$,

$$\begin{aligned} &T_1 = [0, 2h], \quad T_2 = [2h, 3h], \quad x_i = 2h, \quad w_i = T_1 \cup T_2 \\ &\widehat{T}_1 = [-1, 0], \quad \widehat{T}_2 = [0, 1], \qquad\qquad \widehat{w} = \hat{T}_1 \cup \hat{T}_2 \end{aligned}$$

and the affine transformation $F : \hat{\omega} \to \omega_i$. Let $u(x) = x$, $\hat{u} = u \circ F$ and $P_{L^2}\hat{u}$ the local L^2-projection of $\hat{u}$ on $\mathcal{P}_1$, i.e.

$$\int_{\hat{\omega}} (P_{L^2}\hat{u} - \hat{u}) \cdot 1 \,\mathrm{d}t = \int_{\hat{\omega}} (P_{L^2}\hat{u} - \hat{u}) \cdot t \,\mathrm{d}t = 0\,.$$

Show that the local projection error is only of first order, i.e.

$$\frac{\|u - (P_{L^2}\hat{u}) \circ F^{-1}\|_{0,2,\omega}}{\|u\|_{2,2,\omega}} \leq \frac{h}{4\sqrt{3}\,(1 + 3h^2)^{\frac{1}{2}}}\,.$$

Problem 3 (Higher order interpolation estimates)

Let $\mathcal{I}_h : H^{2,2} \to \mathcal{V}_h$ be the Lagrange interpolation, where $d \leq 3$, i.e. $H^{2,2} \hookrightarrow C^0$. Show that for $\varphi \in H^{2,2}(\Omega)$ on $T \in \mathcal{T}_h$, $E \in \mathcal{E}(T)$:

$$\|\varphi - \mathcal{I}_h\varphi\|_{0,2,T} \leq C_1\, h_T^2 |\phi|_{2,2,T}$$
$$\|\varphi - \mathcal{I}_h\varphi\|_{0,2,E} \leq C_2\, h_T^{\frac{3}{2}} |\phi|_{2,2,T}$$

Problem 4 (L^2 a posteriori estimates)

Suppose u and u_h are solutions of $(\tilde{P})$ and $(\tilde{P}_h)$, respectively, with $f, f_h \in L^2(\Omega)$ and $u^\partial = u_h^\partial$. Furthermore, for every $g \in L^2(\Omega)$ there is a solution $\varphi_g \in H^{2,2}(\Omega) \cap \mathring{H}^{1,2}(\Omega)$ of the dual problem

$$a(v, \varphi_g) = (v, g)_{L^2} \quad \forall v \in \mathring{H}^{1,2}(\Omega)$$

with $\|\varphi_g\|_{2,2,\Omega} \leq C\, \|g\|_{0,2,\Omega}$. Show that for the error estimator

$$\eta_T := \Big(h_T^4 \left\|\mathrm{div}(a\nabla u_h) + f_h\right\|_{0,2,T}^2 + \sum_{E\in\mathcal{E}(T)} h_E^3 \left\|[a\nabla u_h \cdot n_E]_E\right\|_{0,2,E}^2 \Big)^{\frac{1}{2}}$$

the following estimate holds:

$$\|u - u_h\|_{0,2,\Omega} \leq C \Big(\|f - f_h\|_{0,2,\Omega} + \big(\sum_{T\in\mathcal{T}_h} \eta_T^2 \big)^{\frac{1}{2}} \Big)$$

Hint: Consider the dual problem with $g := u - u_h$ and make use of problem 3 and the straightforward estimation $\big(\sum_i a_i\big)^2 \leq c \sum_i a_i^2$ for a finite sequence $(a_i)_{i\leq n}$.