Exercise sheet 11.

Closing date **14.1.2014**.

Theoretical exercise 1. (Condition number of the biharmonic equation [5 points])

Consider the biharmonic equation: Find u such that

$$\begin{aligned} \triangle(\triangle u) &= f \qquad \forall x \in \Omega\,, \\ u &= 0 \qquad \forall x \in \partial\Omega\,, \\ \frac{\partial u}{\partial n} &= 0 \qquad \forall x \in \partial\Omega\,. \end{aligned}$$

It can be written in weak form: Find $u \in H_0^2(\Omega) = \{u \in H^2(\Omega) : u|_{\partial\Omega} = 0, \frac{\partial u}{\partial n}|_{\partial\Omega} = 0\}$ such that

$$a(u,v) = F(v) \qquad \forall v \in H_0^2(\Omega)\,.$$

Now assume a square domain $\Omega = (0,1)^2$ and a uniform triangulation with diameter h and local C^1-elements $(\phi_i)_{i=1}^N$. Show that the condition number of the operator matrix $K \in \mathbb{R}^{N\times N}$ with

$$(K)_{ij} = a(\phi_i, \phi_j)$$

grows by $\Theta(h^{-4})$.

Theoretical exercise 2. (Inverse estimate I [5 points])

We have triangle-based quasiuniform triangulations $\{\mathcal{T}_h\}_h$ of a polygonal two-dimensional domain, and we denote the space of linear finite elements by V_h. As usual for quasiuniform triangulations, we have the condition

$$\rho_T \geq \kappa^{-1} h_T \qquad \forall\, T \in \mathcal{T}_h, h > 0\,,$$

where ρ_T denotes the radius of the incircle of T and h_T half of the diameter of T. Show that

$$\|v_h\|_\infty \leq \frac{C}{h}\|v_h\|_0 \qquad \forall\, v_h \in V_h$$

holds with an h-independent constant C.

Theoretical exercise 3. (Inverse estimate II [5 points])

Consider the setup as in Exercise 2. Show that for the discrete solution u_h of the Poisson prolem with homogeneous boundary conditions, the following estimate holds

$$\|u - u_h\|_\infty \leq Ch\|u\|_2\,.$$

Hint: In your proof you need to employ Inverse estimate I.