Scientific Computing I

Winter Semester 2013 / 2014
Prof. Dr. Beuchler
Bastian Bohn and Alexander Hullmann

universität**bonn**

Exercise sheet 12. Closing date **21.1.2014**.

Theoretical exercise 1. (The stokes equations [11 points])

Consider the discrete formulation of the Stokes problem

$$\begin{bmatrix} \mathbf{A} & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$$

with the *vector-Laplacian matrix*

$$(\mathbf{A})_{ij} = \int_\Omega \nabla \vec{\phi}_i : \nabla \vec{\phi}_j$$

and the *divergence matrix*

$$(B)_{kj} = -\int_\Omega \psi_k \nabla \cdot \vec{\phi}_j \ .$$

We have the discrete inf-sup stability condition

$$\gamma \le \min_{q_h \text{ non-constant}} \max_{\vec{v}_h \ne \vec{0}} \frac{|(q_h, \nabla \cdot \vec{v}_h)|}{\|\nabla \vec{v}_h\| \|q_h\|} \ , \tag{1}$$

with $\|\nabla \vec{v}_h\| := \langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle^{1/2}$, $\|q_h\| := \langle Q\mathbf{q}, \mathbf{q} \rangle^{1/2}$ and $|(q_h, \nabla \cdot \vec{v}_h)| = |\langle \mathbf{q}, B\mathbf{v} \rangle|$, where Q is the *pressure mass matrix* with $(Q)_{kl} = \int_\Omega \psi_k \psi_l$. We want to express (1) in terms of the eigenvalues of the *pressure Schur complement* $B\mathbf{A}^{-1}B^T$. To this end, we consider the generalized eigenvalue problem

$$\begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix} = \sigma \begin{bmatrix} \mathbf{A} & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{q} \end{bmatrix} \ , \tag{2}$$

a) [2 points] Show that for non-zero σ, we get $\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle = \langle Q\mathbf{q}, \mathbf{q} \rangle$.

b) [2 points] Show that for non-zero σ, we also get

$$\frac{\langle B\mathbf{A}^{-1}B^T\mathbf{q}, \mathbf{q} \rangle}{\langle Q\mathbf{q}, \mathbf{q} \rangle} = \frac{\langle B^T Q^{-1}B\mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle} = \sigma^2 \ . \tag{3}$$

c) [5 points] Show that (1) implies

$$\gamma^2 = \min_{\mathbf{q} \ne \mathbf{1}} \frac{\langle B\mathbf{A}^{-1}B^T\mathbf{q}, \mathbf{q} \rangle}{\langle Q\mathbf{q}, \mathbf{q} \rangle} \ . \tag{4}$$

d) [2 points] Show that

$$\gamma^2 = \min_{\{\mathbf{v} \in \mathbb{R}^{n_u} : \langle \mathbf{A}\mathbf{v}, \mathbf{u} \rangle = 0, \mathbf{u} \in \ker B\}} \frac{\langle B^T Q^{-1}B\mathbf{v}, \mathbf{v} \rangle}{\langle \mathbf{A}\mathbf{v}, \mathbf{v} \rangle} \ .$$

The closing date for submission of the (bonus) programming exercise is the 28th of January!

Programming exercise 1. (Bonus: Variable functions and reentrant cracks/corners [20 points])

a) [5 points] Use your code to calculate a discrete solution u_h to the problem

$$
\begin{aligned}
-\Delta u(x,y) &= 4 - 2(x^2 + y^2) \quad \text{on } \Omega = (-1,1) \times (-1,1) \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

by using piecewise linear elements. The exact solution to the problem is given by $u(x,y) = (1 - x^2)(1 - y^2)$.

An easy method to employ the varying right hand side in the existing code is to create a function `resetRHS` which is called after the refinement is done. This function then calculates the value of the RHS-function in the barycenter of each element and sets the corresponding material parameter for the elements (Note: you might have to enlarge the `materials` vector of `PDE`). To this end, you can use a function pointer as parameter for `resetRHS` and then just pass the right hand side function.

Use the mesh given in `SampleGrid.txt` (The right hand side still has to be corrected by your code).

Calculate the $L_\infty(\Omega)$ and the normalized ℓ_2 errors on the finite element nodes, i.e. $\sqrt{\frac{1}{n} \sum_{i=1}^{n} (u(x_i, y_i) - u_h(x_i, y_i))^2}$. Here, n is the number of nodes, u is the real solution to the problem and u_h is the discretized solution). Calculate the experimental convergence rates of these errors by computing the solution for different grid refinement levels.

b) [5 points] Calculate the L_2-norm of the error, i.e. $\sqrt{(u - u_h, u - u_h)_{L_2(\Omega)}}$ (It will be useful to create routines for assembling the mass-matrix (without the stiffness part) and for numerical quadrature of the basis functions multiplied with variable coefficients).

c) [5 points] Add Dirichlet zero boundary conditions to the edges 40 and 41 of `SampleGrid.txt`. This can be interpreted as a crack in the underlying material along the positive side of the x-axis. With this modification you are able to solve

$$
\begin{aligned}
-\Delta_{x,y} u(r,\theta) &= (\frac{175}{2}r + 100r^{-\frac{1}{2}} - \frac{375}{2}) \cdot \sin(\frac{1}{2}\theta) \\
&\qquad \text{on } \Omega = (-1,1) \times (-1,1) \setminus \{(x,0) \mid 0 \le x \le 1\} \\
u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

where (r, θ) are the polar coordinates corresponding to the cartesian coordinates (x, y). By $\Delta_{x,y}$ we mean the Laplace operator corresponding to the cartesian coordinates.

The solution can be written as

$$
u(r, \theta) = \begin{cases} (10r^{\frac{1}{2}} - 50r^{\frac{3}{2}} + 50r^2 - 10r^3) \cdot \sin(\frac{1}{2}\theta) & \text{if } r \le 1 \\ 0 & \text{else} \end{cases}
$$

Inspect the convergence rates of the above mentioned errors for this problem.

d) [5 points] In the same way you employed a variable right hand side in the previous subtasks you can also employ varying boundary conditions. Additionally you have to pay attention if you are dealing with Dirichlet or with Neumann boundary conditions

for every specific edge. Implement a `resetBC` function which resets the boundary condition after refinement. Then build a simple triangulation (e.g. divide the unit square into two triangles) and solve the stationary heat equation with the following prescribed temperatures and heat fluxes

$$
\begin{aligned}
\Delta u(x, y) &= 0 & \text{on } \Omega = (0,1) \times (0,1) \\
u(x, y) &= y & \text{for all } (x, y) \in \partial\Omega \cap \{x = 1, 0 \le y \le 1\} \\
u(x, y) &= 0 & \text{for all } (x, y) \in \partial\Omega \cap \{0 \le x \le 1, y = 0\} \\
\frac{\partial u}{\partial \vec{n}} &= -y & \text{for all } (x, y) \in \partial\Omega \cap \{x = 0, 0 < y \le 1\} \\
\frac{\partial u}{\partial \vec{n}} &= x & \text{for all } (x, y) \in \partial\Omega \cap \{0 < x < 1, y = 1\}.
\end{aligned}
$$

It is easy to see that $u(x, y) = xy$ is the exact solution. Calculate the convergence rates for the L_2- and L_∞- errors.

Note that the closing date for submission of the programming exercise is the 28th of January.