Numerical Algorithms

Winter Semester 2015
Prof. Dr. Carsten Burstedde
Jose A. Fonseca

universitätbonn

Exercise Sheet 6.

Due date: **Tuesday, 15.12.15**.

For the initial value problem

$$\begin{cases} q_t + qq_x &= 0, \quad t > 0,\ x \in \mathbb{R}, \\ q(x,0) &= q_0(x), \quad x \in \mathbb{R}. \end{cases} \tag{1}$$

the numerical flux of Godunov's method can be written as $\mathcal{F} = \frac{(Q^\star)^2}{2}$, where $Q^\star = Q^\star(Q_i, Q_j)$ is defined as follows:

If $Q_i \geq Q_j$ then

$$Q^\star := \begin{cases} Q_i, & \text{if } \frac{Q_i + Q_j}{2} > 0, \\ Q_j, & \text{else.} \end{cases} \tag{2}$$

If $Q_i < Q_j$ then

$$Q^\star := \begin{cases} Q_i, & \text{if } Q_i > 0, \\ Q_j, & \text{if } Q_j < 0, \\ 0, & \text{if } Q_i \leq 0 \leq Q_j. \end{cases} \tag{3}$$

Assuming a regular grid in the spacial variable. The CFL condition for this equation imposes that the step size at the time step t^{n+1} must satisfy

$$\Delta t^{n+1} \leq \frac{\Delta x}{\max_i |Q_i^n|} \tag{4}$$

In practice one picks

$$\Delta t^{n+1} = \eta \frac{\Delta x}{\max_i |Q_i^n|} \tag{5}$$

where $0 < \eta \leq 1$ is a safety parameter.

Programming Exercise 1. (Godunov's method for Burgers' equation)

Implement a function `Burgers_solver(x_l, x_r, t_end, eta, p)` that approximates the solution to the Burgers equation using Godunov's method for $x \in$ [`x_l`,`x_r`] and $t \in$ [`0`,`t_end`]. Assume periodic boundary conditions. The size of the time step is determined by (5) on run time. The output of your program should be an animation of the approximate solution as in problem sheet 4.
Test your program using the following initial data functions:

$$u_0(x) = \exp(-4(x-1)^2), \qquad u_0(x) = \begin{cases} 1, & \text{if } \ x \leq 0, \\ 0, & \text{if } \ x > 0, \end{cases} \tag{6}$$

$$u_0(x) = \begin{cases} 0.5, & \text{if } \ x \leq 0, \\ 1, & \text{if } \ x > 0, \end{cases} \qquad u_0(x) = \begin{cases} 1, & \text{if } \ x \leq 0, \\ 1-x, & \text{if } \ 0 < x < 1, \\ 0, & \text{if } \ x \geq 1. \end{cases} \tag{7}$$

(6 points)

Programming Exercise 2. Extend the program coded in the problem sheet 4 such that additionally to Godunov's method the user is able to choose among the following Finite Volume schemes:

- Lax-Friedrichs,
- Lax-Wendroff.

In addition, experiment with other types of boundary conditions, e.g, reflexion waves and inflow boundary conditions. Test out your program to solve the Riemann problem given by the data:

$$A = \begin{pmatrix} 2 & -1 & 1 & 3 \\ -1 & 3 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & 1 \end{pmatrix}, \; q_{\mathrm{l}} = \begin{pmatrix} 1 \\ 1 \\ 0.5 \\ -1 \end{pmatrix} \; q_{\mathrm{r}} = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 1 \end{pmatrix}.$$

for x on the interval $[-4, 4]$ and $t \in [0, 15]$.

(4 points)