Scientific Computing I

Winter semester 2015/2016
Prof. Dr. Marc Alexander Schweitzer
Sa Wu

universitätbonn

Exercise sheet 2.

Submission due **Tue, 2015-11-10, before lecture**.

Exercise 6. (Derivatives of radial functions, polar coordinates)

a) For $a > 0$ consider

$$u : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R} \,, \qquad (x, y) \mapsto \frac{1}{(3x^2 + y^2)^a} \,.$$

Calculate $\frac{\partial}{\partial x} u$, $\frac{\partial^2}{\partial y^2} u$, ∇u, Δu.

b) Show that, using polar coordinates θ, r, the Laplacian $\Delta = \nabla \cdot \nabla$ in $\mathbb{R}^2$ takes the form

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(4 Points)

Exercise 7. (The heat equation on an infinite interval)
On the interval $\mathbb{R}$ a solution $u(x,t) : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ of the heat equation

$$u_t = \frac{\partial}{\partial t} u = \Delta u = u_{xx} \,, \qquad u(x, 0) = u_0(x)$$

for continuous and bounded, but otherwise arbitrary initial value u_0 is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} u_0(y) \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y \,.$$

a) Show that u is smooth (C_∞) for all $t > 0$.

Hint: In this particular case here swapping integration and differentiation is allowed.

b) Show that u indeed solves the heat equation with initial value u_0 in the sense that the equation holds for $t > 0$ and

$$\lim_{(x,t) \to (x_0, 0)} u(x,t) = u_0(x_0) \,,$$

i.e. the continuous extension of u defined by the integral expression to $t = 0$ yields the inital value.

Hint: For an estimate of $|u(x,t) - u_0(x_0)|$ first show that

$$u(x,t) = u_0(x_0) + \int_{\infty}^{\infty} (u_0(y) - u_0(x_0)) \exp\left(-\frac{(x-y)^2}{4t}\right) \mathrm{d}y$$

and split up the remaining integral into integrals over $[x_0 - \delta, x_0 + \delta]$ and $(-\infty, x_0 - \delta) \cup (x_0 + \delta, \infty)$.

(6 Points)

Exercise 8. (Classification of 2nd order partial differential equations)

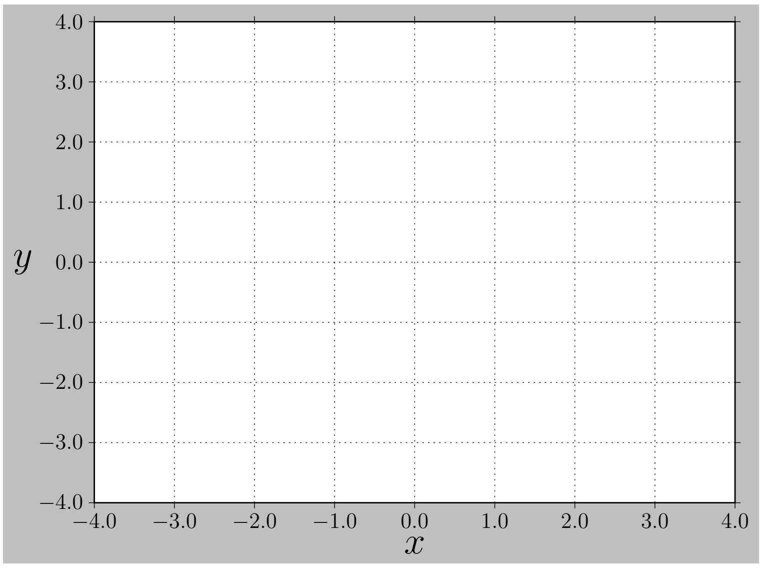
a) Determine the types of the following two partial differential equations.

1) $-4u_{xx} - 3u_{xy} - u_{yx} - u_{yy} + 3u_x - u_y + 2u = 0$
2) $-3u_{xx} + 12u_{xy} + 2u_{yy} - 6u_x + u_y + u = 0$

b) Determine the areas in the the (x, y)-plane in which the equation

$$(1 + x)u_{xx} + 2u_{xy} + yu_{yy} + u_x = 0$$

is elliptic, parabolic, hyperbolic or ultrahyperbolic and, draw them.



(4 Points)

Programming exercise 4. (A phase diagram with `matplotlib`)

a) Write a function to draw the empty grid from exercise 8b) for arbitrary limits and axis labels.

b) Using a), draw the phase diagram from exercise 8b) in `Python/matplotlib`.

(2 Points)