Scientific Computing I

Winter semester 2015/2016
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Exercise sheet 3.

Submission due **Tue, 2015-11-17, before lecture**.

Exercise 9. (PDE type invariance)

We recall that a general scalar PDE of second order can be written as

$$\sum_{i,j=1}^{n} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} u(x) + c(x)u(x) = f(x)$$

with

$$\Omega \ni x = (x_1, \ldots, x_n) , \qquad u \in C^2(\Omega, \mathbb{R}) , \qquad f : \Omega \to \mathbb{R} ,$$
$$(a_{i,j})_{i,j} : \Omega \to \mathbb{R}^{n \times n} , \qquad (b_i)_i : \Omega \to \mathbb{R}^n , \qquad c : \Omega \to \mathbb{R} .$$

Now consider a change of variables $\zeta = \Phi(x) \in C^2(\Omega, \Omega')$ using

$$(\Phi_1, \ldots, \Phi_n) = \Phi , \qquad \forall x \in \Omega : \det D\Phi(x) = \det \left(\frac{\partial}{\partial x_j} \Phi_i(x) \right)_{i,j} \neq 0 .$$

Show that the PDE does not change it's type when being written in new coordinates ζ.
Hint: Chain rule and Sylvester's law of inertia.

(4 Points)

Exercise 10. (An eigenvalue problem)

Consider a thin membrane, circle shaped with radius 1, i.e. $\Omega = B^2 = \{x \in \mathbb{R}^2 : \|x\| = 1\} \subset \mathbb{R}^2$. Assume that this membrane is fixed on the boundary. Computation of eigenfunctions of this membrane leads to the PDE

$$\begin{aligned} \Delta u &= \lambda u & x &\in \Omega = B^2 \\ u &= 0 & x &\in \partial\Omega = S^1 \end{aligned}$$

for some eigenvalue $\lambda \in \mathbb{R}$. Now consider nonzero solutions $u \neq 0$ to this problem.

a) Using a separation of variables $u(r\cos\theta, r\sin\theta) = v(\theta, r) = A(\theta)B(r)$ approach in polar coordinates (θ, r), obtain a system of ordinary differential equations for A, B.

   Hint: Exercise 6b) and $c^2 := -\frac{A''}{A}$.

b) Now assume that one eigenfunction u to some $\lambda \in \mathbb{R}$ were radially symmetric, i.e. $A \equiv$ const.. Using a power series approach

$$B(r) = \sum_{k=0}^{\infty} b_k r^k$$

   determine the first 4 terms of the series and a corresponding approximation of λ.

   Note: This is by no means a complete analysis of the eigenvalues of the problem. Here, we are only investigating eigenvalues of radially symmetric eigenfunctions.

(5 Points)

Exercise 11. (Free fall)

An approximation of the equations of motion of a body in free fall with air resistance are given by

$$mv'(t) = \frac{1}{2}\rho C A v^2(t) - mg \qquad v(0) = 0$$

with v being the body's velocity, m its mass, $g = 9.81ms^{-2}$ being acceleration due to gravity, $\rho = 1.2kgm^{-3}$ the density of air, $C = 0.5$ the drag coefficient depending on the geometry of the body, A the body's cross sectional area.

a) State the dimensions of the quantities appearing in the the equation and, show that the equation is dimensionally homogeneous.

b) Without additional boundary conditions the falling body will asymptotically approach a limiting velocity U as $t \to \infty$. Using the asymptotic series expansion $v = \sum_{k=0}^{\infty} a_k t^{-k}$, calculate U.

c) Show that using

$$\tilde{t} = \frac{t}{\tau} \qquad V(\tilde{t}) = \frac{v(t)}{U}$$

   for some appropriate τ yields the nondimensionalized form

$$V'(\tilde{t}) = 1 - V^2(\tilde{t}) \qquad V(0) = 0$$

   of the equation of motion of the falling body and, compute τ.

(4 Points)

Exercise 12. (Linear system from central differences)

Consider the first order boundary value problem

$$y'(x) = f(x) \text{ for } x \in (a,b), \qquad y(a) = \alpha\ , \qquad y(b) = \beta\ .$$

We are now attempting to compute approximations $y_i \approx y(x_i)$ with $x_i = a + ih$, $i = 1, 2, \ldots, n-1$, $n > 2$ and $h := \frac{b-a}{n}$ using central differences $y'(x_i) \approx \frac{1}{2h}(y_{i+1} - y_{i-1})$.

a) Determine the linear system

$$\underbrace{(a_{i,j})_{i,j}}_{=A}\underbrace{(y_j)_j}_{=y} = \underbrace{(f_i)_i}_{=f}$$

corresponding to the boundary value problem.

b) Show that A is singular for even n.

(3 Points)