Scientific Computing I

Winter semester 2015/2016
Prof. Dr. Marc Alexander Schweitzer
Sa Wu

universitätbonn

Exercise sheet 4.

Submission due **Tue, 2015-11-24, before lecture**.

Exercise 13. (Stability and consistency for the transport equation)
Consider the linear transport equation

$$\begin{aligned} (Pu)(x,t) = \partial_t u(x,t) + b\partial_x u(x,t) &= 0\ , & (x,t) &\in (0,1)\times(0,T)\ , \\ u(x,0) &= g(x)\ , & x &\in (0,1)\ . \end{aligned}$$

Now assume that finite difference discretizations $u_j^n = u(x_j, t_n)$, $(Du)_j^n = 0$, D being the form the equation takes when using discrete operators, fulfilling the CFL condition

$$|r| := \left| b\frac{\Delta t}{\Delta x} \right| \leq 1$$

are given. Further assume that the initial condition g admits a solution $u \in \mathcal{C}^4$.
Recall that a discretization u_j^n for the transport equation is called stable if

$$K^n := \max \left| u_j^n \right| \leq K^0 := \max \left| g(x_j) \right|\ .$$

Also, recall that the consistency error of this discretization is given by

$$\tau_j^n = (Pu)(x_j, t_n) - (Du)_j^n)\ .$$

Remark: In some literature the term *conditionally stable* (i.e. depending on some condition, here CFL) is used. Stability can also be shown with *von Neumann stability analysis*, i.e. checking boundedness of the *amplification factor* $G = \max|u_j^{n+1}/u_j^n| \overset{!}{\leq} 1$ for $u(x,t) = \exp(at + ibx)$ with $a, b \in \mathbb{R}$.

a) Show that the Friedrich method

$$u_j^{n+1} = \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right) - \frac{r}{2}\left(u_{j+1}^n - u_{j-1}^n\right)$$

is stable and has consistency order 1 in time 2 in space, i.e.

$$\left|\tau_j^n\right| \in \mathcal{O}\left(\Delta t + \Delta x^2\right)\ .$$

b) Show that the Lax-Wendroff method

$$u_j^{n+1} = u_j^n - \frac{r}{2}\left(u_{j+1}^n + u_{j-1}^n\right) + \frac{r^2}{2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right)$$

is stable and has consistency order 2.

Hint: Show that $G^2 = 1 - (r^2 - r^4)(1 - \cos(b\Delta x))^2$.

(6 Points)

Programming exercise 5. (Upwind scheme for the transport equation)

Consider the case of the transport equation used for the introduction of the upwind scheme

$$u_j^{n+1} = u_j^n + b\frac{\Delta t}{\Delta x}\left(u_j^n - u_{j-1}^n\right)$$

with $b = 1$, $u_j^n \approx u(x_j, t_n)$, $\Delta x = \frac{1}{M}$, $x_j = j\Delta x$, $j = 0, \ldots, M$, $\Delta t = \frac{1}{N}$, $t_n = n\Delta t$, $j = 0, \ldots, N$, for the initial boundary value problem

$$\begin{aligned} \partial_t u + b\partial_x u &= 0 & (x,t) &\in (0,1)\times(0,1) \\ u(0,t) &= 0 & t &> 0 \\ u(x,0) &= g(x) & x &\in (0,1)\ . \end{aligned}$$

Let $a = 0.2$ and

$$g_1 = \begin{cases} 1 & x \in (0,2a) \\ 0 & \text{otherwise} \end{cases}, \qquad g_2 = \begin{cases} \exp\left(-\frac{1}{1-\left(\frac{x-a}{a}\right)^2}\right) & x \in (0,2a) \\ 0 & \text{otherwise} \end{cases}.$$

a) Implement the upwind scheme.

b) Plot the exact solutions to initial data g_1, g_2 at times $t = 0, 0.25, 0.5$ into one figure for each g.

c) For $M = N = 2^4, 2^8$, plot the numerical solutions obtained with the upwind scheme to initial data g_1, g_2 at times $t = 0, 0.25, 0.5$ into one figure for each g.

(3 Points)

Exercise 14. (Consistency estimates for functions with insufficient regularity)

For $0 < h < 1$ consider the difference operators ∂^+, ∂^-, ∂^0 and $\partial^+\partial^-$ applied to the function

$$u : [0,2] \to \mathbb{R} \qquad x \mapsto \frac{1}{6}\begin{cases} -(x-1)^3 & \text{for } 0 \le x \le 1 \\ (x-1)^3 & \text{for } 1 < x \le 2 \end{cases}.$$

Compute the errors of the difference operators in $x = 1$. Compare these results with the theoretical results known from the lecture.

(3 Points)

Exercise 15. (Difference operators for non-uniform mesh widths)
Let h_W, h_E, h_N and h_S be arbitrary positive numbers.

a) For $x \in \mathbb{R}$ and $u \in \mathbb{C}^3(\mathbb{R})$ let

$$u_W := u(x - h_W) \qquad u_Z := u(x) \qquad u_E := u(x + h_E).$$

Show that

$$u_{xx}(x) = \frac{2}{h_E(h_E + h_W)} u_E + \frac{2}{h_W(h_E + h_W)} u_W - \frac{2}{h_E h_W} u_Z + \mathcal{O}(h)$$

with $h := \max\{h_W, h_E\}$. What happens for $h_E = h_W$?

b) For $(x, y) \in \mathbb{R}^2$ and $u \in \mathbb{C}^3(\mathbb{R}^2)$ let

$$\begin{aligned} u_N &:= u(x, y + h_N) \\ u_W := u(x - h_W, y) \qquad u_Z &:= u(x, y) \qquad u_E := u(x + h_E, y) \\ u_S &:= u(x, y - h_S) \end{aligned}$$

Show that

$$\begin{aligned} (\Delta u)(x) = &\frac{2}{h_E(h_E + h_W)} u_E + \frac{2}{h_W(h_E + h_W)} u_W + \frac{2}{h_S(h_S + h_N)} u_S \\ &+ \frac{2}{h_N(h_S + h_N)} u_N - \left(\frac{2}{h_E h_W} + \frac{2}{h_S h_N} \right) u_Z + \mathcal{O}(h) \end{aligned}$$

with $h := \max\{h_W, h_E, h_N, h_S\}$. What happens for $h = h_E = h_W = h_N = h_S$?

(4 Points)