Scientific Computing I

Winter semester 2015/2016
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Exercise sheet 5.

Submission due **Tue, 2015-12-01, before lecture**.

Exercise 16. (Construction of Finite Differences)

Let $h > 0$ be sufficiently small and let all considered functions $u : \mathbb{R}^2 \supseteq \Omega \to \mathbb{R}$ be sufficiently smooth. Given a specific finite difference approximation of some derivative, the consistency order of said finite difference is obtained via Taylor expansion. We can turn this procedure around and construct finite differences of given order from appropriate Taylor expansions.
We recall the notation

$$\begin{bmatrix} \alpha_{-1,+1} & \alpha_{0,+1} & \alpha_{+1,+1} \\ \alpha_{-1,0} & \alpha_{0,0} & \alpha_{+1,0} \\ \alpha_{-1,-1} & \alpha_{0,-1} & \alpha_{+1,-1} \end{bmatrix} u(x,y) = \sum_{i,j\in\{-1,0,+1\}} \alpha_{i,j} u(x+ih, y+jh)$$

for a compact 9 punkt stencil representing a finite difference using $u(x+ih, y+ih)$, $i,j \in \{-1,0,1\}$.
In the following exercises, use this 9 punkt stencil.

a) Construct a finite difference of consistency order 2 for the approximation of $\partial_x \partial_y u$.

b) Show that no compact 9 punkt stencil for the Laplacian Δ has consistency order 3 or greater.

(4 Points)

Exercise 17. (Smoothing properties of the heat equation)

Consider the initial boundary value problem

$$\begin{aligned} \partial_t u &= \partial_x^2 u \,, & (x,t) &\in (0,\pi) \times (0,\infty) \,, \\ u(0,t) = u(\pi,t) &= 0 \,, & t &\in [0,\infty) \,, \\ u(x,0) &= u_0(x) \,. \end{aligned}$$

a) Using a separation of variables approach, obtain

$$u(x,t) = \sum_{k=1}^{\infty} c_k \exp(-k^2 t) \sin(kx) \qquad c_k = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin(kx) \mathrm{d}x$$

Hint: Exercise 10a) and the $\exp(-\lambda x)$ approach for linear ordinary differential equations with constant coefficients.

b) Show that $\|\partial_t u(\cdot,t)\|^2_{L^2(0,\pi)} = \frac{\pi}{2} \sum_{k=1}^{\infty} c_k^2 k^4 \exp(-2k^2 t)$.

c) Let $\sigma > 0$. Show that there exists $M > 0$ which depends only on σ such that $\|\partial_t u\|_{L^2(0,\pi)} \le M$ for all $t \ge \sigma$.

(4 Points)

Exercise 18. (Discretizations of the heat equation)

Let $\mu = \frac{\Delta t}{\Delta x^2}$. For the following discretizations of the homogeneous heat equation

$$\partial_t u = \partial_{xx} u$$

obtain the consistency order. Moreover, check if these schemes are stable.
Hint: For stability assume a solution $u(x,t) = A^t \exp(ibx)$ and show $|A| \leq 1$.

a) Leapfrog

$$u_j^{n+1} - u_j^{n-1} - 2\mu\big(u_{j-1}^n + u_{j+1}^n\big) + 4\mu u_j^n = 0$$

b) du-Fort-Frankel

$$u_j^{n+1} - u_j^{n-1} - 2\mu\Big(u_{j-1}^n - \Big(u_j^{n+1} + u_j^{n-1}\Big) + u_{j+1}^n\Big) = 0$$

(4 Points)

Exercise 19. (Green's identity)

Let $\Omega \subset \mathbb{R}^2$ be a simply connected smooth domain with smooth boundary $\partial\Omega$. Let $u, \sigma \in \mathbb{C}^2(\Omega)$ and $v \in \mathbb{C}^1(\Omega)$.

a) Show that the first Green's identity

$$\int_\Omega \Delta u v \mathrm{d}x = \int_{\partial\Omega} v \nabla u \cdot n \mathrm{d}S - \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x$$

holds.

b) Show that

$$\int_\Omega \nabla \cdot (\sigma \nabla u)\, v \mathrm{d}x = \int_{\partial\Omega} v\sigma \nabla u \cdot n \mathrm{d}S - \int_\Omega \sigma \nabla u \cdot \nabla v \mathrm{d}x\ .$$

Hint: Gauss's divergence theorem.

(4 Points)