Scientific Computing I

Winter semester 2015/2016
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Exercise sheet 6.

Submission due **Tue, 2015-12-08, before lecture**.

Exercise 20. (The d'Alembert solution of the wave equation)

In the lecture it was mentioned that the one-dimensional wave equation

$$\begin{aligned} \partial_t^2 u(x,t) &= c^2 \partial_x^2 u(x,t) & (x,t) &\in \mathbb{R} \times (0,\infty) \\ u(x,0) &= u_0(x) & x &\in \mathbb{R} \\ \partial_t u(x,0) &= v_0(x) & x &\in \mathbb{R} \end{aligned}$$

for $c > 0$,$u \in C^2(\mathbb{R} \times (0,\infty), \mathbb{R})$, $u_0 \in C^2(\mathbb{R})$, $v_0 \in C^1(\mathbb{R})$ has the d'Alembert solution

$$u(x,t) = \frac{1}{2}(u_0(x+ct) + u_0(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(y)\mathrm{d}y \; .$$

We will now examine this solution in more detail.

a) Use a change of variables $\xi = x - ct, \eta = x + ct, v(\xi,\eta) = u(x,t)$ to show that the wave equation reduces to

$$\partial_\xi \partial_\eta v = 0 \; .$$

Follow that $u(x,t) = v(\xi,\eta) = f(\xi) + g(\eta) = f(x-ct) + g(x+ct)$ for some functions f, g.

b) Use the boundary conditions $u(x,0) = u_0(x), \partial_t u(x,0) = v_0(x)$ to solve for f, g and obtain the d'Alembert solution.

c) Verify that the d'Alembert solution is actually a valid solution to the given initial value problem.

(6 Points)

Programming exercise 6. (An explicit discretization of the wave equation)

The lecture introduced the discretization

$$
\begin{aligned}
u_j^{n+1} &= 2(1-\mu^2)u_j^n + \mu^2\big(u_{j-1}^n + u_{j+1}^n\big) - u_j^{n-1} && n \in \{1,\dots,N-1\}, j \in \{1,\dots,M-1\}\\
u_j^0 &= u_0(x_j) && j \in \{1,\dots,M-1\}\\
u_j^1 &= (1-\mu^2)u_j^0 + \frac{\mu^2}{2}\big(u_{j-1}^0 + u_{j+1}^0\big) + \Delta t v_0(x_j) && j \in \{1,\dots,M-1\}\\
u_0^n = u_M^n &= 0 && n \in \{0,\dots,N\}
\end{aligned}
$$

with $\Delta x = \frac{1}{M}$, $x_j = j\Delta x$, $j = 0,\dots,M$, $\Delta t = \frac{1}{N}$, $t_n = n\Delta t$, $n = 0,\dots,N$, $\mu = c\frac{\Delta t}{\Delta x}$, $u_j^n \approx u(x_j, t_n)$ for the wave equation

$$
\begin{aligned}
\partial_{tt}u(x,t) - c^2\partial_{xx}u(x,t) &= 0 && (x,t) \in (0,1)\times(0,T)\\
u(x,0) &= u_0(x) && x \in (0,1)\\
\partial_t u(x,0) &= v_0(x) && x \in (0,1)\\
u(0,t) = u(1,t) &= 0 && t \in (0,T)
\end{aligned}
$$

on the bounded interval $(0,1)$.

a) Show that the discretization for u_j^{n+1} with $n \in \{1,\dots N-1\}$ is stable given the CFL condition $\mu^2 \leq 1$. *Hint*: Exercise 18, $\cos x = 1 - 2\sin^2(\frac{x}{2})$.

b) Implement this discretization in Python for

$$
c = 2\,, \qquad T = 2\,, \qquad M = 2^6\,,
$$

and, choose N, and thus Δt, to fulfill the CFL condition.

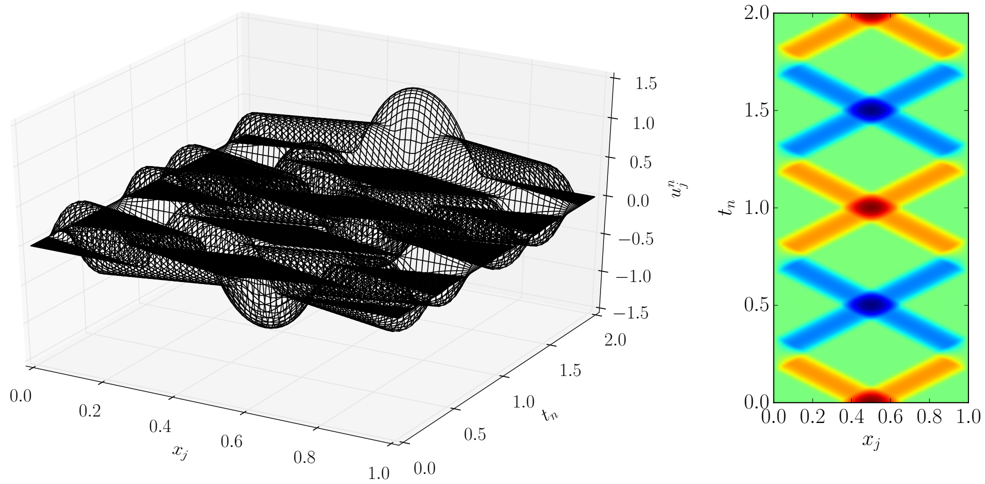
c) Produce insightful plots of the solution u_j^n for initial values

$$
u_0 = \begin{cases} \exp\left(1 - \frac{1}{1-\left(\frac{x-.5}{.2}\right)^2}\right) & |x-0.5| \leq 0.2\\ 0 & \text{otherwise} \end{cases} \qquad v_0 = 0
$$

and for initial values

$$
u_0 = 0 \qquad v_0 = 1
$$

Hint: `imshow` and `mplot3d`, in particular `plot_wireframe`.



(6 Points)

Exercise 21. (Comparison principle)

Let Ω be a bounded domain. Let $u_1, u_2 \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be harmonic with boundary values

$$u_1|_{\partial\Omega} = g_1 , \qquad u_2|_{\partial\Omega} = g_2 , \qquad g_1|_{\partial\Omega} \leq g_2|_{\partial\Omega} .$$

a) Show that

$$\forall x \in \Omega : \qquad u_1(x) \leq u_2(x) .$$

b) Let Ω be connected and $g_1(y) < g_2(y)$ for some $y \in \partial\Omega$. Show that

$$\forall x \in \Omega : \qquad u_1(x) < u_2(x) .$$

(4 Points)