# Scientific Computing I

Winter semester 2015/2016
Prof. Dr. Marc Alexander Schweitzer
Sa Wu

universitätbonn

## Exercise sheet 8.

Submission due **Tue, 2016-01-12, before lecture**.

The lecture on Tuesday, 2015-12-22 will be held. There will be no tutorials on Wednesday 2015-12-23 through Friday 2016-01-08. The next tutorials after the break take place on Wednesday, 2016-01-13 and Friday, 2016-01-15.

**Exercise 26.** (Compact embeddings)

A theorem from the lecture where the proof is left as an exercise is the following. Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces. Show the following.

a) If $X \hookrightarrow Y$, there exists $C > 0$ such that $\forall \varphi \in X : \|\varphi\|_Y \leq C\|\varphi\|_X$.

b) $X \overset{C}{\hookrightarrow} Y \Rightarrow X \hookrightarrow Y$.

c) If $X \overset{C}{\hookrightarrow} Y$ and $(\varphi_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ is a bounded sequence in $X$, then $(\varphi_n)_{n\in\mathbb{N}} \in Y^{\mathbb{N}}$ has a convergent subsequence in $Y$.

(4 Points )

**Exercise 27.** (Classical solutions to elliptic problems)

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $A : \Omega \to \mathbb{R}^{d\times d}$ be symmetric positive definite for all $x$, $a_0$ be nonnegative, $f : \Omega \to \mathbb{R}$ be continuous. Consider the linear space

$$V := \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$$

Show that

$$a(u, v) := \int_\Omega (A(x)\nabla u(x)) \cdot \nabla v(x) + a_0(x)u(x)v(x)\mathrm{d}x$$

is a symmetric, positive definite bilinear form.

(4 Points )

**Exercise 28.** (Sobolev spaces)

Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}$ and $1 \leq p < \infty$. Complete the theorem from the lecture by showing the following.

a) $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ is a Banach space.

*Hint*: For the $\Delta$ inequality use the $(l^p(N), \|\cdot\|_p)$ inequality.

b) $(H^k(\Omega), \langle\cdot,\cdot\rangle_{H^k(\Omega)})$ is a Hilbert space.

c) Let $\xi \in C_0^\infty(\Omega)$, $u \in W^{k,p}(\Omega)$. Then $\xi u \in W^{k,p}(\Omega)$.

(4 Points )

**Exercise 29.** (Denseness of $C^\infty$ in $W^{k,p}$)

Let $\Omega \subset \mathbb{R}^d$ be open, $k \in \mathbb{N}$ and $1 \leq p < \infty$. The denseness of $C^\infty$ in $W^{k,p}$ requires some more steps.
Recall that

$$L^p_{\text{loc}}(\Omega) := \{u : \Omega \to \mathbb{R} : \forall K \subseteq \Omega \text{ compact} : u|_K \in L^p(K)\}$$
$$W^{k,p}_{\text{loc}}(\Omega) := \{u : \forall |\alpha| \leq k : D^\alpha u \in L^p_{\text{loc}}(\Omega)\}$$

Let $\eta_\epsilon$ be a mollifier as in exercise 24 and let $f_\epsilon = \eta_\epsilon * f$.

a) For $f \in L^p_{\text{loc}}$, show that $f_\epsilon \to f$ in $L^p_{\text{loc}}$ for $\epsilon \to 0$, i.e. $\forall K \subseteq \Omega$ compact $\|f_\epsilon - f\|_{L^p(K)} \to 0$ for $\epsilon \to 0$.

b) Follow that for $f \in W^{k,p}_{\text{loc}}$ we similarly have $f_\epsilon \to f$ in $W^{k,p}_{\text{loc}}$.

c) Consider

$$U_i := \{x \in \Omega : \text{dist}(x_i, \partial\Omega) > \frac{1}{i}, \|x\| < i\} \ , \quad V_0 := U_2 \ , \quad V_i := U_{i+3} \setminus \overline{U_{i+1}} \ , i \geq 0$$

and a smooth partition of unity $\xi_i$ subordinate to the cover $V_i$, i.e.

$$\xi_i \in C^\infty \ , \qquad \operatorname{supp} \xi_i \subseteq V_i \ , \qquad 0 \leq \xi_i \leq 1 \ , \qquad \sum_{i=0}^{\infty} \xi_i \equiv 1 \ .$$

Let $u \in W^{k,p}$, $\epsilon > 0$. Using b), show that there exist $\epsilon_i > 0$ such that

$$\|\eta_{\epsilon_i} * (\xi_i u) - (\xi_i u)\|_{k,p} \leq \frac{\epsilon}{2^{i+1}}$$

d) Let $v = \sum_{i=0}^{\infty} \eta_{\epsilon_i} * (\xi_i u)$. Using the definition of the $\xi_i$, show that for compact $V \subseteq \Omega$

$$\|u - v\|_{W^{k,p}(V)} \leq \epsilon \ .$$

and follow that $C^\infty$ lies dense in $W^{k,p}$.

(8 Points )

**Programming exercise 7.** (Finite Difference discretizations of the Poisson equation) Consider discretizations of the Poisson equation with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \qquad && x \in \Omega\ , \\ u &= g && x \in \partial\Omega \end{aligned}$$

with the five point stencil for $\Omega \subset \mathbb{R}^2$.
For the following cases, determine and plot the (nonzero) structure of the matrix $L_h$.
*Remark*: With `matplotlib` consider using `spy`.

a) The lecture case with $\Omega = (0,1)^2$, $h = \frac{1}{n}$, $(x_i, y_j) = (ih, jh)$, $i, j = 0, \dots, n$ and lexicographical linear ordering of gridpoints $(i,j) \mapsto j(n+1) + i$.

b) $\Omega := (0,a) \times (0,b)$ with $a = nh$, $b = mh$ and lexicographical ordering.

c) b with a so called chess ordering. This ordering first takes the lexicographical ordering of all black checkers, i.e. $u_{i,j}$ with even $i + j$, and the white checkers, i.e. $i + j$ odd, afterwards.

d) $\Omega := B_1(0) = \{x \in \mathbb{R}^2 \|x\|_2 \leq 1\}$ with lexicographical ordering of the interior points given as $(x_i, y_j) = (-1 + ih, -1 + jh)$ with $x_i^2 + y_j^2 < 1$.

For the case of $B_1(0)$ we are going to implement the Shortley-Weller approximation at the boundary, i.e. taking the discretization from exercise 15 for nodes close to the boundary. That is for $(x_i, y_j)$ with $x_i^2 + x_j^2 < 1$ but $x_(i+k)^2 + y_(j+l)^2 > 1$ with $k, l \in \{-1, 0, 1\}$ replace the missing nodes from the 5 point stencil with $x_i + k\alpha_{i,j}h, y_j + l\beta_{i,j}h$ with $\alpha_{i,j}, \beta_{i,j} \in [0,1]$ lying on the boundary $\partial\Omega$.

a) Write Python functions to determine $(x_i, y_j) \in \Omega$ and assemble $L_h, K_h, f_h, g_h$ for given $n$.

b) Write a Python function to numerically solve the Poisson equation on $B_1(0)$ for an exact solution

$$u = e^x \sin y$$

with $f, g$ to be computed from $u$.

c) Check the convergence properties with a convergence plots. Extrapolate the order of convergence.

d) Check the stability of the discretization by plotting $\|L_h^{-1}\|$.

(12 Points )