Scientific Computing I

Winter semester 2015/2016
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Exercise sheet 9.

Submission due **Tue, 2016-01-19, before lecture**.

Exercise 30. (Weak derivatives)

In the lecture we already saw that functions with singularities may be contained in the Sobolev spaces $W^{k,p}$ for higher dimensions d. We consider another example of this phenomenon.
Let

$$\Omega := B_1(0) = \{x \in \mathbb{R}^2 \,:\, \|x\| < 1\} \text{ and } \qquad u(x) := \log\log\left(\frac{2}{\|x\|}\right) .$$

Show that $u \in H^1(\Omega)$.
Hint: Consider polar coordinates, as in the lecture for $\|x\|^s$, and use upper bounds for $\log(y), y > 0$.

(4 Points)

Exercise 31. (Variational formulation of a PDE)

Let Ω be a bounded domain. Recall the PDE from the lecture

$$\begin{aligned} -\nabla \cdot A(x)\nabla u(x) + a_0(x)u(x) &= f(x) && x \in \Omega \\ u(x) &= g(x) && x \in \Gamma_D \subset \partial\Omega \end{aligned}$$

with $A(x)$ being symmetric, positive definite, bounded, a_0 non-negative and bounded, and Γ_D having nonzero area.
As mentioned in the lecture, we can restrict ourselves to the homogeneous case $g = 0$. Thus, we need only consider solutions

$$u \in V = \{u \in C^2(\Omega) \cap C^0(\overline{\Omega}) : u|_{\Gamma_D} = 0\} .$$

Show that u satisfies a variational formulation

$$a(u, v) = l(v) \qquad \forall v \in V$$

with

a) l linear and bounded,

b) a symmetric, bilinear and positive.

(4 Points)

Exercise 32. (A minimization problem)

Let Ω be a bounded domain with sufficiently smooth boundary and $g : \partial\Omega \to \mathbb{R}$ be a given function. We are looking for the function $u \in H^1(\Omega)$ minimizing the H^1 norm and coinciding with g on the boundary, i.e.

$$u = \operatorname*{argmin}_{v \in H^1 : v|_{\partial\Omega} = \gamma(v) = g} \|v\|_{H^1} .$$

Under what conditions can this problem be handled with a variational formulation?

(2 Points)

Exercise 33. (The pure Neumann problem)

Consider the pure Neumann problem

$$\begin{aligned} -\nabla \cdot A(x)\nabla u(x) + a_0(x)u(x) &= f(x) \qquad && x \in \Omega \\ (A(x)\nabla u(x)) \cdot \nu(x) &= h(x) && x \in \partial\Omega \end{aligned}$$

with outer normal $\nu(x)$ and A, a_0 as above.
Show that f and h have to satisfy a compatibility condition for the existence of a solution.

(2 Points)

Exercise 34. (Application of the trace theorem)

In the lecture it was mentioned that for a Lipschitz domain Ω and $l \leq k-1$ there exists a continuous trace operator

$$\begin{aligned} \gamma_l : H^k(\Omega) &\to L^2(\Omega) && \text{such that} \\ \gamma_l(\phi) &= \left(\frac{\partial}{\partial \nu}\right)^l \phi|_{\partial\Omega} \qquad && \forall \phi \in C^k(\overline{\Omega}) . \end{aligned}$$

Assume further that Ω has a piecewise smooth boundary and let $u \in H^1 \cap C(\overline{\Omega})$. Show that $u \in H^1_0$ and $u|_{\partial\Omega}$ are equivalent.

(4 Points)