Exercise sheet 10.

Submission due **Tue, 2016-01-26, before lecture**.

Exercise 35. (A boundary value problem without solution)

On the interval $[0,1]$ consider the bilinear form

$$a(u,v) := \int_0^1 x^2 u'(x) v'(x) x.$$

a) What is the boundary value problem corresponding to the variational problem

$$\frac{1}{2} a(u,u) - \int_0^1 u(x) \mathrm{d}x \overset{!}{=} \min \; ?$$

b) Show that this problem does not have a solution in $H_0^1((0,1))$.

(3 Points)

Exercise 36. (H^1 and H^0)

a) Exhibit a function $u \in C^1([0,1])$ which is not contained in $H^1((0,1))$.

b) In the case of $k = 0$ we have $H_0^0(\Omega) = H^0(\Omega)$. Exhibit a sequence $u_n \in (C_0^\infty)^{\mathbb{N}}$ which converges to the constant function $v = 1$ in the L^2 sense.

(3 Points)

Exercise 37. (Embeddings of l_p)

Consider the spaces $l_p \subseteq \mathbb{R}^{\mathbb{N}}$ of infinite sequences satisfying the condition

$$\|x\|_p := \|x\|_{l_p} := \|(x_k)_k\|_{l_p} := \left(\sum_{k \in \mathbb{N}} |x_k|^p \right)^{\frac{1}{p}} .$$

Recall that the spaces $(l_p, \|\cdot\|_p)$ are Banach spaces.

a) Show that $l_1 \hookrightarrow l_2$.

b) Is this embedding compact?

(3 Points)

Exercise 38. (Weak divergence)

Let $\Omega \subset \mathbb{R}^d$ a bounded domain with sufficiently smooth boundary. For vector fields $u = (u_1, \ldots, u_d) : \Omega \to \mathbb{R}^d$ we have, similar to the scalar case, the spaces $H^1(\Omega)^d$ as completions of the infinitely often differentiable vector fields $C^\infty(\Omega)^d$ with respect to the norm

$$\|u\|^2_{H^1(\Omega)^d} = \sum_{k=1}^{d} \|u_k\|^2_{H^1(\Omega)} .$$

For PDEs involving only $\operatorname{div} u := \nabla \cdot u := \sum_{k=1}^{d} \partial_k u_k$ it might be worthwhile to consider functions with only sufficient regularity for div.
Consider the weak divergence $w = \nabla \cdot u$ defined through

$$\int_\Omega w\varphi = -\int_\Omega u \cdot \nabla\varphi \quad \forall \varphi \in C_0^\infty(\Omega)$$

and the corresponding space $H(\operatorname{div}, \Omega)$ as the completion of $C^\infty(\Omega)^d$ with respect to the norm

$$\|u\|^2_{H(\operatorname{div},\Omega)} := \|u\|^2_{L^2(\Omega)^d} + \|\operatorname{div} u\|^2_{L^2(\Omega)} .$$

a) First, we check that the definition for the weak divergence makes sense. For $u : \Omega \to \mathbb{R}^d, \varphi : \Omega \to \mathbb{R}$ sufficiently smooth show that

$$\int_\Omega \varphi \nabla \cdot u \mathrm{d}x = -\int_\Omega \nabla\varphi \cdot v \mathrm{d}x + \int_{\partial\Omega} \varphi u \cdot \nu \mathrm{d}s .$$

b) Show that $H^1(\Omega)^d \subset H(\operatorname{div}, \Omega) \subset L_2(\Omega)^d$.

c) Show that a piecewise polynomial v is contained in $H(\operatorname{div}, \Omega)$ if and only if the components in the direction of normals $v \cdot \nu$ are continuous on the interelement boundaries.

   Hint: a) and the theorem on $\varphi \in H^k(\Omega) \Leftrightarrow \varphi \in C^{k-1}(\Omega)$ from the lecture.

(4 Points)

Exercise 39. (Affine transformations)

In this exercise we complete a Lemma on polynomial basis functions on triangles from the lecture. Show the following.

a) The degree of a polynomial is invariant under affine transformations.

b) The image of a triangle under affine transformations is a triangle.

(3 Points)