An Optimal Adaptive Finite Element Method for an Obstacle Problem

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1 Introduction

- $K := \{v \in V | \chi \leq v \text{ a.e. in } \Omega\}$
- $a(v, w) := \int_\Omega \nabla v \cdot \nabla w dx$ for all $v, w \in V$
- $\|\!| \cdot |\!\| := a(\cdot, \cdot)^{1/2}$
- $F := (f, \cdot)_{L^2(\Omega)}$
- $u \in K$ unique solution of $F(v - u) \leq a(u, v - u)$ for all $v \in K$
- $u_l \in K(T_l)$ unique solution of $F(v_l - u_l) \leq a(u_l, v_l - u_l)$ for all $v_l \in K_l$

2 Algorithm and Main Results

2.1 Adaptive Algorithm

- $\theta \eta_l^2 \leq \eta_l^2(\mathcal{M}_l) := \sum_{E \in \mathcal{M}_l} \eta_l^2(E)$ (bulk criterion)

2.2 Main Results

Theorem 2.1. *The respective solutions u_l and u_{l+m} to the discrete problem*

$$F(v_l - u_l) \leq a(u_l, v_l - u_l) \text{ for all } v_l \in K_l$$

with respect to the triangulation T_l and its refinement T_{l+m} satisfy

$$E(u_l) - E(u_{l+m}) + \|\!| u_{l+m} - u_l |\!\|^2 \leq C_{dRel} \eta_l^2(M_{l,l+m})$$

for some subset $\mathcal{M}_{l,l+m}$ of $\mathcal{E}_l$ with $|\mathcal{M}_{l,l+m}| \preceq |\mathcal{T}_l \backslash \mathcal{T}_{l+m}|$

Theorem 2.2. *Suppose $(u, f) \in \mathcal{A}_s$ for some $s > 0$ and $\theta < c_{Eff}/(C_{dRel}+1)$. Then the output $(\mathcal{T}_l, V_l, u_l)_{l \in \mathbb{N}}$ of the adaptive algorithm of Section 2.1 satisfies*

$$\delta_l + \gamma \eta_l^2 \preceq |(u, f)|^2_{\mathcal{A}_s} (|\mathcal{T}_l| - |\mathcal{T}_0|)^{-2s} \text{ for all } l = 1, 2, \ldots$$

3 Proof of Discrete Reliability

3.1 $\rho(z) := F(\varphi_z^{(l)}) - a(u_l, \varphi_z^{(l)}) \leq 0 \leq u_l(z) - \chi(z)$ for any $z \in \mathcal{N}_l(\Omega)$

3.2 $\mathcal{C}_l := \{z \in \mathcal{N}_l(\Omega) : u_l(z) = \chi(z)\}$

3.3 LHS $= \sum_{z \in \mathcal{N}_l} F((e - e_z)\varphi_z^{(l)}) + \sum_{z \in \mathcal{N}_l(\Omega)} e_z \rho_z - a(u_l, e - e_l)$

3.4 $-a(u_l, e - e_l) \preceq \eta_l(\mathcal{E}_l \backslash \mathcal{E}_{l+m}) \|e\|$

3.5 control of $F((e - e_z)\varphi_z^{(l)}) + e_z \rho_z$ in the case $z \in \mathcal{C}_l$ with $u_l \equiv \chi$ on $\omega_z^{(l)}$

- $(f, \varphi_z^{(l)}(e - e_z))_{L^2(\omega_z^{(l)})} + e_z\rho_z \preceq osc(f, \omega_z^{(l)}) \|e\|_{\omega_z^{(l)}}$

3.6 $\sum_{z \in \mathcal{N}_l(\partial\Omega) \cap \{\chi < 0\}} F((e - e_z)\varphi_z^{(l)}) \preceq Osc_l \|e\|$

3.7 case $u_l \neq \chi$ on $\omega_z^{(l)}$ for $z \in \mathcal{N}_l(\Omega)$ (other case already discussed in 3.5)

- $\|h_l f\|^2_{L^2(\omega_z^{(l)})} \preceq \sum_{y \in \mathcal{N}_l(\omega_z^{(l)})} (Osc^2(f, \omega_y^{(l)}) + \eta_l^2(\mathcal{E}_l(y)))$

3.8 Upper bound for $F((e - e_z)\varphi_z^{(l)}$ in 4 cases

3.9 We decomposite $\mathcal{N}_l(\Omega) = \mathcal{U}_l \cup \mathcal{I}_l \cup \mathcal{R}_l$
  - $\mathcal{U}_l := \{z \in \mathcal{N}_l(\Omega) : \mathcal{T}_l(z) = \mathcal{T}_{l+m}(z)\}$ (neighborhood unrefined)
  - $\mathcal{I}_l := \mathcal{N}_l(\Omega) \backslash (\mathcal{U}_l \cup \mathcal{R}_l)$ (intermdiate refinement)
  - $\mathcal{R}_l := \{z \in \mathcal{N}_l(\Omega) : \mathcal{T}_l(z) \subset \mathcal{T}_l \backslash \mathcal{T}_{l+m}\}$ (neighborhood totally refined)

- Now we have reduced patches $\omega_z^* := \omega_z^{(l)} \backslash \Omega^{‘}$ where $\Omega^{‘} := \bigcup_{z \in \mathcal{U}_l} \omega_z^{(l)}$

3.10 $min_{g \in P_1(\omega_z^{(l)})} \|v_l - g\|_{L^2(\omega_z^{(l)})} \approx \|v_l\|_{L^2(\omega_z^{(l)})}$ where $v_l \in P_1(\mathcal{T}_l(z)) \cap C8\omega_z^{(l)})$ with $v_l(z) = 0$,$0 \leq v_l$ on $\omega_z^{(l)}$

3.11 Search upper bound of $e_z\rho_z$ in the remaining case $u_l \neq \chi$ on $\omega_z^{(l)}$ while $u_l(z) = \chi(z)$

- $e_z\rho_z \preceq (\eta_l(\mathcal{E}_l(z)) + \sum_{y \in \mathcal{N}_l(\overline{\omega_z^{(l)}})} Osc(f, \omega_y^{(l)}))(\|e\|_{\omega_z^{(l)}} + \eta_l(\mathcal{E}_l(z)))$

3.12 $\mathcal{M}_{l,l+m} := \{E \in \mathcal{E}_l(\Omega) : \exists F \in \mathcal{E}_l \exists G \in \mathcal{E}_l \backslash \mathcal{E}_{l+m}$ such that $E \cap F \neq \emptyset \neq F \cap G\}$

- $LHS \preceq (\eta_l(\mathcal{M}_{l,l+m}) + Osc_l(\mathcal{M}_{l,l+m}))(\|e\| + \eta_l(\mathcal{M}_{l,l+m}))$

4 Optimal Convergence Rates

4.1 Efficiency of the Error Estimator

Lemma 4.1 (Efficiency). *There exists some* $c_{Eff} = 1$ *with*

$$c_{Eff}\eta_l^2 \leq \delta_l + Osc_l^2$$

4.2 Contraction Property

Theorem 4.1. *There exist constants* $\gamma > 0$ *and* $0 < q < 1$ *such that*

$$\delta_{l+1} + \gamma\eta_{l+1}^2 \leq q(\delta_l + \gamma\eta_l^2) \text{ for all } l = 0, 1, 2, \ldots$$

Lemma 4.2. *Let* $\mathcal{T}_{l+m}$ *be some refinement of* $\mathcal{T}_l$ *with*

$$\delta_{l+m} + Osc_{l+m}^2 \leq q(\delta_l + Osc_l^2)$$

for some $0 < q < 1$*. Then it holds*

$$c_{Eff}(1 - q)\eta_l^2 \leq (1 + C_{dRel})\eta_l^2(\mathcal{M}_{l,l+m})$$

4.3 Optimality

- $\mathbb{E}(\mathcal{T}_0, N; u, f) := inf_{\mathcal{T} \in \mathbb{T}(\mathcal{T}_0, N)} min_{v_\mathcal{T} \in K(\mathcal{T})} (E(v_\mathcal{T}) - E(u) + Osc_\mathcal{T}^2)$

Lemma 4.3. *Suppose that* $(u, f) \in K \times L^2(\Omega)$ *satisfies* $|(u, f)|_{\mathcal{A}_s} < \infty$ *for some* $0 < s < \infty$ *and suppose that* $0 < \theta < c_{Eff}/(C_{dRel} + 1)$ *from Lemma 4.3 (here 4.2). Then,*

$$|\mathcal{M}_l|^2 \preceq |(u, f)|_{\mathcal{A}_s}^{2/s} (\delta_l + Osc_l^2)^{-1/s} \text{ for all } l = 0, 1, 2, \ldots$$