Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Problem sheet 1

Please hand in the solutions in the exercise classes on Wednesday/Thursday November 2/3!

Exercise 1 **4 Points**

Let $\Omega \subset \mathbb{R}^2$ a polygonal domain and let $\mathcal{T} = (T_i)_{i=1,\ldots,I}$ be a triangulation of Ω. Prove that for a continuous function $u_h : \Omega \to \mathbb{R}$, which is piecewise affine w.r.t. $\mathcal{T}$ (i.e. u_h is affine on each T_i), the surface area of the graph of u_h is given by

$$\mathcal{E}[u_h] = \sum_{i=1}^{I} |T_i| \sqrt{1 + |\nabla u_h|_{T_i}|^2}\,.$$

Exercise 2 **4 Points**

Let $\Omega \subset \mathbb{R}^n$ and $p > 1$. Consider the energy

$$\mathcal{E}[u] = \int_\Omega |\nabla u|^p \mathrm{d}x\,.$$

(i) Compute the derivate $D\mathcal{E}[u](v) := \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{E}[u + sv]\big|_{s=0}$ for a test function v. What are suitable function spaces for u and v?
(ii) Derive from $D\mathcal{E}[u](v) = 0$ a partial differential equation for u.

Exercise 3 **4 Points**

For any $N \geq 2$ and $h = \frac{1}{N}$ let

$$\begin{aligned} x_i &= i \cdot h\,, && i = 0, \ldots, N\,, \\ x_{i+\frac{1}{3}} &= x_i + \tfrac{1}{3}h\,, && i = 0, \ldots, N-1\,, \\ x_{i+\frac{2}{3}} &= x_i + \tfrac{2}{3}h\,, && i = 0, \ldots, N-1\,. \end{aligned}$$

Consider the finite element space

$$\mathcal{V}_h^3([0,1]) = \left\{ v \in C^0([0,1], \mathbb{R}) \;:\; v\big|_{[x_i, x_{i+1}]} \in \mathcal{P}_3([x_i, x_{i+1}]) \;\forall i = 0, \ldots, N-1 \right\}\,.$$

Here, $\mathcal{P}_3([x_i, x_{i+1}])$ is the set of all cubic polynomials on $[x_i, x_{i+1}]$. Compute the set of base functions $(\phi_h^i)_{i=0,\ldots,N}\,, (\phi_h^{i+\frac{1}{3}})_{i=0,\ldots,N-1}\,, (\phi_h^{i+\frac{2}{3}})_{i=0,\ldots,N-1} \subset \mathcal{V}_h^3([0,1])$, which satisfy $\phi_h^{i+\frac{k}{3}}(x_{j+\frac{l}{3}}) = \delta_{i,j}\delta_{k,l}$ for $k, l \in \{0, 1, 2\}$.

Hint: Split the construction of the base functions on the different intervals $[x_j, x_{j+1}]$.

Exercise 4 **4 Points**

Let $a : (0,1) \to \mathbb{R}$ be the following function:

$$a(x) = \begin{cases} c_1 & \text{if } x \in (0, b_1)\,, \\ c_2 & \text{if } x \in [b_1, b_2)\,, \\ c_3 & \text{if } x \in [b_2, 1)\,, \end{cases}$$

for $c_1, c_2, c_3 \in \mathbb{R}$ and $0 < b_1 < b_2 < 1$. Determine a weak solution $u \in H^1(0,1) = W^{1,2}(0,1)$ of the boundary value problem

$$\begin{aligned} -(a(x)u'(x))' &= 0 \quad \forall x \in (0,1)\,, \\ u(0) &= 0\,, \\ u(1) &= 1\,. \end{aligned}$$