Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn

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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Prof. Dr. Martin Rumpf
Alexander Effland — Stefanie Heyden — Stefan Simon — Sascha Tölkes

Problem sheet 10

Please hand in the solutions on Tuesday January 17!

Exercise 31 **2+4 Points**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Consider the Neumann boundary problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega\,, \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega \end{aligned}$$

where $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$. We additionally require $\int_\Omega u \,\mathrm{d}x = 0$ to impose uniqueness of the solution since $u + c$ for a $c \in \mathbb{R}$ is a solution if u solves this boundary problem.

(i.) Show that

$$\int_\Omega f \,\mathrm{d}x + \int_{\partial\Omega} g \,\mathrm{d}a = 0$$

is a necessary condition for the solvability of the Neumann boundary problem.

(ii.) Prove the existence of a weak solution of the Neumann boundary problem. To this end, derive the saddle point formulation and show ellipticity of a suitable bilinear form and prove an inf-sup-condition.

Exercise 32 **2+4 Points**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Consider the biharmonic equation

$$\begin{aligned} -\Delta^2 u &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \\ \Delta u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $f \in L^2(\Omega)$.

(i.) Use $w = \Delta u$ to formulate a saddle point formulation for the biharmonic equation. Choose u and w in $H_0^1(\Omega)$ and prove the inf-sup-condition.
(ii.) Derive a finite element formulation for the saddle-point problem and prove the discrete inf-sup-condition and V_h-ellipticity.

Exercise 33 **4+4 Points**

(i.) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Prove that $H(\mathrm{div})$ is a Hilbert space with inner product

$$(u, v)_{H(\mathrm{div})} := (u, v)_{L^2(\Omega,\mathbb{R}^n)} + (\mathrm{div} u, \mathrm{div} v)_{L^2(\Omega,\mathbb{R})}$$

for any $u, v \in H(\mathrm{div})$.
Remark: The particular task is the proof of the completeness of this space.
(ii.) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary and $\mathcal{T}_h$ a triangulation on Ω. The Banach space $H(\mathrm{rot})$ is defined as the completion of $C^\infty(\Omega, \mathbb{R}^2)$ w.r.t. the norm

$$\|f\|_{H(\mathrm{rot})}^2 = \|f\|_{0,2,\Omega}^2 + \|\mathrm{rot} f\|_{0,2,\Omega}^2 ,$$

where $\mathrm{rot} f = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}$. Define

$$\mathcal{S}_h = \left\{ p \in L^2(\Omega, \mathbb{R}^2) : p\big|_T \in \mathcal{P}_m(T) \text{ for } m \in \mathbb{N} \text{ and } T \in \mathcal{T}_h \right\} .$$

Let $f \in \mathcal{S}_h$. Show that $f \in H(\mathrm{rot})$ if for all interior edges $E \in \mathcal{E}_h$ the function $f \cdot \tau_E$ is continuous on E, where τ_E is the tangent vector of E.

Exercise 34 **6 Points**

Let $m, n \geq 1$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with polygonal boundary and $\mathcal{T}_h$ be any triangulation on Ω. Prove that for any $f \in W^{m,p}(T)$ for an element $T \in \mathcal{T}_h$ a polynomial $g \in \mathcal{P}_{m-1}(T)$ exists such that for $k \in \{0, \ldots, m\}$

$$|f - g|_{W^{k,p}(T)} \leq C h^{m-k} |f|_{W^{m,p}(T)}$$

for a positive constant C depending solely on m and Ω.
Hint: Follow the proof of the approximation error estimate for the Lagrange interpolation.