Institut für Numerische Simulation
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# Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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## Problem sheet 13

## Chapter 0

- Define the variational formulation of Poisson's problem and of minimal graph surfaces.
- Derive the Euler-Lagrange equations in both cases.
- How to discretize these equations with finite elements?

## Chapter 1

- When does a function have weak derivatives?
- Define Sobolev spaces $H^{m,p}(\Omega)$.
- Explain the trace theorem.
- Explain the Poincaré estimate (both versions) and prove the one for zero boundary data.
- Explain how to prove existence of solutions using the Riesz theorem and the Lax Milgram theorem and compare both.
- Relate the $L$-condition to a weak formulation of an elliptic PDE.
- Explain how to treat inhomogeneous boundary data.
- Compute a solution for a system with jumping coefficients in 1D.
- Give the definition of a general finite element and explain some examples of Lagrange and Hermite finite elements.
- Why does the definition of affine equivalence perfectly fit to the interpolation theory in Sobolev spaces?
- Explain the general procedure to get interpolation estimates.
- What is the role of norm equivalence in finite dimensions in this case?

- Show different ways to assemble the linear system of equations resulting from a finite element discretization.
- How to incorporate inhomogeneous boundary data in the implementation?
- State and prove Céa's lemma and prove convergence of the finite element approximation.
- Explain the Aubin-Nitzsche trick to prove $L^2$ estimates.

## Chapter 2

- Explain the bisection algorithm.
- Explain the general procedure to get residual error estimates.
- Prove the a posteriori estimates for Poisson's problem and Lagrange finite elements.
- Discuss the construction of test functions to prove efficiency.
- Why do we need a weak interpolation approach?
- Sketch the proof of the interpolation result.
- Derive the local estimates used in the a posteriori theory.
- Discuss the problem of guaranteed convergence.
- Explain the structure of the proof of convergence of the adaptive algorithm for Poisson's problem.

## Chapter 3

- Explain the two versions of Strang's lemma.
- Discuss the Crouzeix-Raviart finite element and explain how to prove convergence.

## Chapter 4

- Explain the saddle point problem underlying a mixed formulation.
- Compare this approach for Poisson's problem and for Stokes equation.
- What are the assumptions to prove existence of a saddle point?
- Describe and compare the continuous and weak $\inf - \sup$ condition and explain Fortin's theorem in this context.
- Compare the formulation using bilinear forms and the one for operators.
- What is the role of the closed range theorem in this context?
- Describe the two versions of a mixed formulation of Poisson's problem.

- When and in which sense are the operator $B$ and $B^*$ invertible?
- Prove existence of a saddle point and the invertibility of the $L$ operator.
- Explain how to prove convergence of discrete solutions to continuous solutions.
- Why is the $\inf - \sup$ condition fulfilled for the $H(div)$ formulation of Poisson's problem?
- Discuss the Raviart-Thomas element and explain the actual degrees of freedom.
- Explain how to solve the discrete divergence equation.
- What error estimate do we get?
- Explain the Stokes system of equations.
- State the incompressible Navier-Stokes equation.
- Show how to prove existence of solutions for the stationary Stokes equation with Dirichlet boundary condition.
- Explain what is meant by spurious pressure for badly chosen discretization.
- Explain the MINI element and the Taylor-Hood element.
- How to show the discrete $\inf - \sup$ condition in both cases?
- What convergence results do we obtain?
- Explain the penalty method and the Schur complement approach.
- What is known about the condition number of the Schur complement matrix?
- Discuss the computational complexity of both approaches.

**Remark:** Elasticity theory and approximation of elastic deformations will also be part of the exam. To which extend will be discussed next week.