Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

Winter 2016/17

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Problem sheet 2

Please hand in the solutions in the lecture on Tuesday November 8!

Remark: Exercise 4 was also contained in sheet 1. It is suggested to hand in the solution of this exercise with the other solutions of this sheet on November 8!

Exercise 4 — 4 Points

Let $a : (0,1) \to \mathbb{R}$ be the following function:

$$a(x) = \begin{cases} c_1 & \text{if } x \in (0, b_1)\,, \\ c_2 & \text{if } x \in [b_1, b_2)\,, \\ c_3 & \text{if } x \in [b_2, 1)\,, \end{cases}$$

for $c_1, c_2, c_3 \in \mathbb{R}$ and $0 < b_1 < b_2 < 1$. Determine a weak solution $u \in H^{1,2}(0,1)$ of the boundary value problem

$$\begin{aligned} -(a(x)u'(x))' &= 0 \quad \forall x \in (0,1)\,, \\ u(0) &= 0\,, \\ u(1) &= 1\,. \end{aligned}$$

Exercise 5 — 4 Points

Consider the function

$$u_n \in H^{1,2}((0,1), \mathbb{R})\,, \quad x \mapsto \frac{1}{4\sqrt{\frac{1}{4} + \frac{1}{n^2}}} - \frac{(x - \frac{1}{2})^2}{\sqrt{(x - \frac{1}{2})^2 + \frac{1}{n^2}}}\,.$$

Show that u_n converges w.r.t. the $H^{1,2}((0,1))$-norm to

$$u \in H^{1,2}((0,1), \mathbb{R})\,, \quad x \mapsto \frac{1}{2} - \left| x - \frac{1}{2} \right|\,.$$

Recall that for any $f \in H^{1,2}((0,1))$ the $H^{1,2}((0,1))$-norm is defined as

$$\|f\|_{1,2}^2 = \int_0^1 (f(x))^2 \,\mathrm{d}x + \int_0^1 (f'(x))^2 \,\mathrm{d}x\,.$$

Exercise 6 **4 Points**

Let $\Omega \subset \mathbb{R}^2$ be a open and bounded domain with smooth boundary. Let $\Gamma \subset \Omega$ be an injective smooth curve that separates Ω into two non-empty and connected sets Ω^+ and Ω^-. Let $f \in C^1(\overline{\Omega})$ be such that $f\big|_{\Omega^+} \in C^\infty(\Omega^+)$ and $f\big|_{\Omega^-} \in C^\infty(\Omega^-)$. Show that f is twice weakly differentiable.

Exercise 7 **4 Points**

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and $a_{ij}, b_i, b \in L^\infty(\Omega)$ with

$$\sum_{i,j=1}^{n} a_{i,j}\xi_i\xi_j \geq c_0|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ with $c_0 > 0$. Verify that the following operators are bounded and coercive bilinear forms on $H_0^{1,2}(\Omega)$:

i) $L_1(u,v) := \int_\Omega \sum_{i,j} a_{i,j}\partial_i u \partial_j v \, + buv \, \mathrm{d}x$
under the assumption $c_0 > \|b\|_\infty C_P(\Omega)$.

ii) $L_2(u,v) := \int_\Omega \sum_{i,j} a_{i,j}\partial_i u \partial_j v \, + \sum_i b_i \partial_i uv \, \mathrm{d}x$
under the assumption $c_0 > \frac{1}{2}\left(\sum_i \|b_i\|_\infty (1 + C_P(\Omega))\right)$.

Hints:
Use the Poincaré inequality $\|u\|_2^2 \leq C_P(\Omega)\|\nabla u\|_2^2$ with a constant $C_p(\Omega)$ depending solely on Ω.

For ii) use $\partial_i u \cdot u \leq \frac{1}{2}(\partial_i u)^2 + \frac{1}{2}u^2$.