Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Problem sheet 4

Please hand in the solutions on Tuesday November 22!

Exercise 10 **4 Points**

Consider the following generalized definition :

Definition (General Finite Element)**.** Let

1. $K \subset \mathbb{R}^n$ be a bounded closed set, $K \neq \emptyset$, with piecewise smooth boundary,
2. $\mathcal{P}$ be a k-dimensional space of functions on K ($k \geq 1$),
3. the set of degrees of freedom $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ be a basis for $\mathcal{P}'$.

Then $(K, \mathcal{P}, \Gamma)$ defines a *finite element*.

Let $\mathcal{Q}_k = \{\sum_j c_j p_j(x) q_j(y) : p_j, q_j \in \mathcal{P}_k\}$ and K be a rectangle.

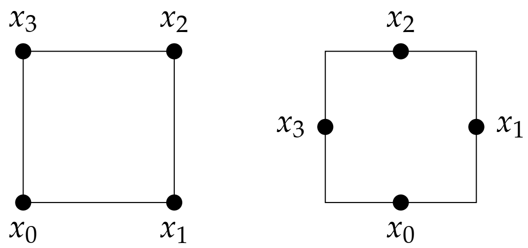


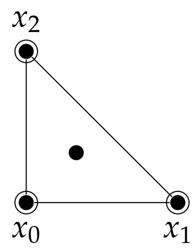
Figure 1: Left: bilinear Lagrange element. Right: no finite element.

Here, a filled point indicates that at this vertex the value is a degree of freedom.
(i.) Show that $(K, \mathcal{Q}_1, \Gamma)$ with Γ as depicted in the left drawing in Figure 1 is a finite element (*bilinear Lagrange element*).
(ii.) Show that $(K, \mathcal{Q}_1, \Gamma)$ with Γ as depicted in the right drawing in Figure 1 is *no* finite element.

Exercise 11 **4 Points**

Consider the Hermite finite element $(T, \mathcal{P}_3, \Gamma)$ with the following 10 degrees of freedom

$$\Gamma(p) = (\Gamma_\alpha(p))_{\alpha=1,\ldots,10} = \left\{ p(x_i), \partial_k p(x_i), p\left(\frac{x_0+x_1+x_2}{3}\right) \right\}_{i\in\{0,1,2\},k\in\{1,2\}}.$$



(i.) Show that any function in $\mathcal{P}_3(T)$ is uniquely determined by an Hermite finite element function on T.
(ii.) Let $\mathcal{T}_h$ be any triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$, $\mathcal{V}_h$ be the space of Hermite finite elements on $\mathcal{T}_h$. Show that a function $v \in \mathcal{V}_h$ is not necessarily differentiable.

Exercise 12 **4 Points**

Consider the quartic finite element $(T, \mathcal{P}_4, \Gamma)$ with the following 15 degrees of freedom

$$\Gamma(p) = (\Gamma_\alpha(p))_{\alpha=1,\ldots,15} = \left\{ p(x_i), p(x_j), \partial_k p(x_i), D^2 p(x_i)(x_l - x_i, x_m - x_i) \right\}$$

for $i, l, m \in \{0,1,2\}, j \in \{3,4,5\}, k \in \{1,2\}, i \neq m \neq l \neq i$.

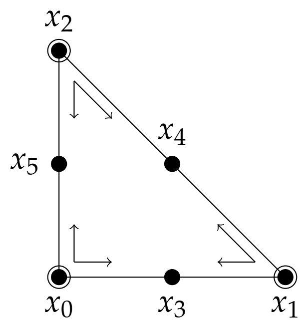


Figure 2: Quartic finite element.

Here, a filled point/circle indicates that at this vertex or midpoint the value/the gradient is given. An arrow shows that at the vertex the second order directional derivative in the direction of the line segments is provided.
(i.) Prove that any function in $\mathcal{P}_4(T)$ is uniquely determined by this finite element function on the triangle T.
(ii.) Show that $(T, \mathcal{P}_4, \Gamma)$ is affine equivalent to the finite element $(\hat{T}, \mathcal{P}_4, \hat{\Gamma})$ on the reference triangle $\hat{T}$ with nodes $\hat{x}_0 = (0,0)$, $\hat{x}_1 = (1,0)$, and $\hat{x}_2 = (0,1)$.

Exercise 13 **4 Points**

Let $\omega \in (\pi, 2\pi)$ and

$$\Omega = \{ \ (r\cos(\varphi), r\sin(\varphi)) \quad | \quad 0 < r < 1, \quad 0 < \varphi < \omega \ \}.$$

Further, let u be the solution of

$$\begin{aligned} -\Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega\,, \end{aligned}$$

where $g(r, \varphi) = r^{\frac{\pi}{\omega}} \sin(\frac{\pi}{\omega}\varphi)$.
Now, consider a uniform triangulation of Ω and the finite element space $\mathcal{V}_h^1(\Omega)$ of continuous and piecewise linear finite elements. Prove the estimate

$$\|u - I_h u\|_{H^{1,2}(\Omega)} \leq C(u, \omega) h^{\frac{\pi}{\omega}},$$

where I_h is the interpolation operator on $\mathcal{V}_h^1(\Omega)$.
Hint: Use that for a suitable $\alpha = \alpha(h) > 0$ we have that $u \in H^{2,2}(\Omega \setminus B_\alpha(0))$. Further, estimate the $H^{1,2}$-norm on $B_\alpha(0)$.