Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Problem sheet 7

Please hand in the solutions on Tuesday December 13!

Exercise 21 **6 Points**

Let $\Omega \subset \mathbb{R}^d$ be a polygonal domain with $d \leq 3$ and $\mathcal{T}_h$ be a regular triangulation of Ω. Further, let $V_h \subset H_0^{1,2}(\Omega)$ be a finite element space with $P_1 \subset \hat{P}$, s.t. all elements are affine equivalent to a reference element, and $\mathcal{I}_h$ be the Lagrange interpolation operator. Consider the bilinear form $a : H_0^{1,2} \times H_0^{1,2} \to \mathbb{R}$, $a(w,v) = \int_\Omega \nabla w \cdot \nabla v \,\mathrm{d}x$. For $f, f_h \in L^2(\Omega)$ let $u \in H_0^{1,2}(\Omega)$ be the solution of $a(u,v) = \int_\Omega f \cdot v \,\mathrm{d}x$ for all $v \in H_0^{1,2}(\Omega)$, and $u_h \in V_h$ be the discrete solution of $a(u_h, v_h) = \int_\Omega f_h \cdot v_h \,\mathrm{d}x$ for all $v_h \in V_h$.

Prove the a posteriori error estimate

$$\|u - u_h\|_{0,2,\Omega} \leq C\|f - f_h\|_{0,2,\Omega} + C\left(\sum_{T \in \mathcal{T}_h} \mu_T^2\right)^{\frac{1}{2}},$$

where

$$\mu_T^2 := \|h(T)^2(f_h + \Delta u_h)\|_{0,2,T}^2 + \sum_{E \in \mathcal{E}_0(T)} \|h(T)^{\frac{3}{2}} \left[\nabla u_h \cdot n\right]_E \|_{0,2,E}^2 \,.$$

Hint: Consider for $w = u - u_h$ a dual solution φ_w s.t.
$a(v, \varphi_w) = \int_\Omega wv \,\mathrm{d}x$ for all $v \in H_0^{1,2}(\Omega)$.
Make use of the regularity estimate $\|\varphi_w\|_{2,2,\Omega} \leq c\|w\|_{0,2,\Omega}$ and the interpolation estimates $\|\varphi_w - \mathcal{I}_h\varphi_w\|_{0,2,T} \leq ch(T)^2\|\varphi_w\|_{2,2,T}$ and $\|\varphi_w - \mathcal{I}_h\varphi_w\|_{0,2,E} \leq ch(T)^{\frac{3}{2}}\|\varphi_w\|_{2,2,T}$ (see Exercise 19).

Exercise 22 **4+2+4=10 Points**

Let $I = (0,1)$ be the unit interval. Consider the bilinear form $a(u,v) = \int_I u'v' \,\mathrm{d}x$.

(i) Let for $x \in I$ the Dirac distribution δ_x defined by $\langle \delta_x, \phi \rangle := \int_I \phi(y) \,\mathrm{d}\delta_x(y) := \phi(x)$ for all $\phi \in C_c^\infty(I)$.
Find a function $G_x \in H_0^1(I)$ s.t.

$$a(v, G_x) = \int_I v(y) \,\mathrm{d}\delta_x(y) \qquad \text{for all } v \in H_0^1(I) \,.$$

Hint: Consider the intervals $(0,x)$ and $(x,1)$ and make the ansatz that G_x is piece-wise smooth.

(ii) Let $\mathcal{T}_h$ be an equidistant mesh on I with grid size h. Furthermore, let V_h be the space of piece-wise affine and continuous finite elements w.r.t. $\mathcal{T}_h$, and let $\mathcal{I}_h$ be the corresponding Lagrange interpolation operator. Prove the following estimate for the function G_x:

$$\|G_x - \mathcal{I}_h G_x\|_{0,1,I} \leq ch^2 \,.$$

Attention: Take into account that in general x is not a multiple of h.

(iii) For $f, f_h \in L^2(I)$ let $u \in H_0^{1,2}(I)$ be the solution of $a(u,v) = \int_I f \cdot v \,\mathrm{d}x$ for all $v \in H_0^{1,2}(I)$, and $u_h \in V_h$ be the discrete solution of $a(u_h, v_h) = \int_I f_h \cdot v_h \,\mathrm{d}x$ for all $v_h \in V_h$.
Prove the a posteriori error estimate

$$\|u - u_h\|_{0,\infty,I} \leq C\|f - f_h\|_{0,\infty,I} + Ch^2 \left(\sum_{T \in \mathcal{T}_h} \|(f_h + \Delta u_h)\|_{0,\infty,T} \right) \,.$$

Hint: Use that $a(u - u_h, G_x) = (u - u_h)(x)$.