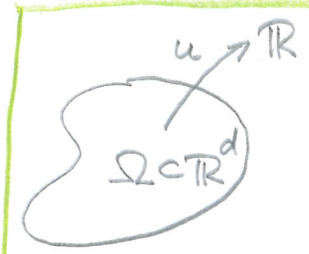
Lecture Course — Scientific Computing I

topics

- problems from the calculus of variations
- Finite Element discretization
- inexact solution and Strangs lemma
- a priori & a posteriori error analysis
- non conforming finite elements
- elasticity, thin plates, thin shells

0. Review of the basics of Finite Element theory

(E)



$d = 1,2,3$

$$\mathcal{E}[u] = \int_\Omega W(x, u(x), \nabla u(x))\, dx$$

Find $u : \overline{\Omega} \to \mathbb{R}$, which minizes $\mathcal{E}[\cdot]$ subject to some boundary condition

$$u(x) = u^\partial(x) \quad \forall x \in \partial\Omega \quad (u^\partial : \overline{\Omega} \to \mathbb{R})$$

0.1. Examples

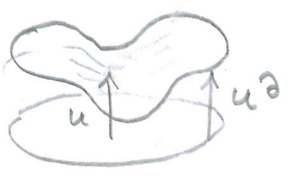
(i) $$\mathcal{E}[u] = \frac{1}{2}\int_\Omega a(x)\nabla u(x) \cdot \nabla u(x) - f(x)\,u(x)\, dx$$

$a : \Omega \to \mathbb{R}^{d,d}$ (e.g. diffusivity tensor $\rightsquigarrow$ $\mathcal{E}$ enthalpy of a heat conducting object in equilibrium configuration)

simplest case: $a(x) = \mathbb{1}$

$$\mathcal{E}[u] = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx \qquad \text{(Dirichlet energy)}$$

(ii) Minimal surface graphs



$$\mathcal{E}[u] = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\, dx$$

$\hookrightarrow$ area of the graph surface

0.2. Future extensions:

- vector valued pbs $\rightsquigarrow$ elasticity
- non flat domains $\rightsquigarrow$ deformations of thin shells

At first we ask for a necessary condition for u to be a minimizer of the energy $\mathcal{E}[\cdot]$:

$$\underbrace{\frac{d}{ds}\mathcal{E}[u+s\varphi]\Big|_{s=0} = 0}_{\text{"directional" derivative of } \mathcal{E}[\cdot] \ (1^{st} \text{ variation of } \mathcal{E})} \qquad \forall \varphi : \bar{\Omega} \to \mathbb{R}, \quad \varphi|_{\partial\Omega} = 0$$

$$\frac{d}{ds}\int_\Omega W(x, u(x)+s\varphi(x), \nabla u(x) + s\nabla\varphi(x))\,dx\,\Big|_{s=0}$$

$$\boxed{W : \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\ ; \ (x,u,p) \mapsto W(x,u,p)}$$

$$= \int_\Omega W_{,u}(x,u(x),\nabla u(x))\,\varphi(x) + \underbrace{\sum_{i=1}^d W_{,p_i}(x,u(x),\nabla u(x))\,\partial_i\varphi(x)}_{W_{,p}(x,u(x),\nabla u(x))\cdot\nabla\varphi(x)}\,dx$$

necessary condition:

$$(\tilde{P}) \quad \boxed{0 = \int_\Omega W_{,u}(x,u,\nabla u)\,\varphi + W_{,p}(x,u,\nabla u)\cdot\nabla\varphi\,dx \qquad \forall\varphi : \bar{\Omega}\to\mathbb{R},\ \varphi|_{\partial\Omega} = 0}$$

example (i): $(x,u,p) = \frac{1}{2}a(x)\,p\cdot p - f(x)\,u$

$a : \Omega \to \mathbb{R}^{d,d}$, $f : \Omega \to \mathbb{R}$

$$\Rightarrow (\tilde{P})\ 0 = \int_\Omega f(x)\,\varphi(x) + a(x)\nabla u(x)\cdot\nabla\varphi(x)\,dx$$

(ii) $W(x,u,p) = \sqrt{1+|p|^2}$

$$\Rightarrow W_{,p}(x,u,p) = \frac{p}{\sqrt{1+|p|^2}}$$

$$\Rightarrow \quad (\tilde{P}) \qquad 0 = \int_\Omega \frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}} \cdot \nabla \varphi(x)\, dx$$

Recall: Gauß-Theorem $\displaystyle\int_\Omega \operatorname{div} v(x)\, dx = \int_{\partial\Omega} v(x)\cdot n(x)\, da$

application: $v(x) = \psi(x)\varphi(x)\, e_j$ for $j \in \{1, \ldots d\}$

$\psi, \varphi : \overline{\Omega} \to \mathbb{R}$

$$\underbrace{\int_\Omega \operatorname{div} v(x)\, dx}_{= \int_\Omega \partial_j \psi(x)\varphi(x) + \psi(x)\partial_j\varphi(x)\, dx} = \underbrace{\int_{\partial\Omega} v(x)\cdot n(x)\, da}_{= \int_{\partial\Omega} \psi(x)\varphi(x)\, n_j(x)\, da}$$

$$\Rightarrow \boxed{\int_\Omega \partial_j \psi(x)\,\varphi(x)\, dx = -\int_\Omega \psi(x)\,\partial_j\varphi(x)\, dx + \int_{\partial\Omega} \psi(x)\varphi(x)\, n_j(x)\, da}$$

(integration by parts)

analogously: $w : \overline{\Omega} \to \mathbb{R}^d$, $\varphi : \overline{\Omega} \to \mathbb{R}$

$v = w\varphi$

$$\int_\Omega \operatorname{div}(w\varphi)\, dx = \int_\Omega \varphi\, w\cdot n\, da \quad \Rightarrow$$

$$\boxed{\int_\Omega w\cdot\nabla\varphi\, dx = -\int_\Omega \operatorname{div} w\, \varphi\, dx + \int_{\partial\Omega} \varphi\, w\cdot n\, da}$$

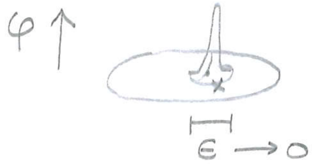
Now we use this for our necessary conditions to be fulfilled by a minimizing u $(w = W_{,p}(x,u,\nabla u))$:

$$0 = \int_\Omega W_{,u}(x,u(x),\nabla u(x))\varphi(x) - \underbrace{\operatorname{div} W_{,p}(x,u(x),\nabla u(x))}_{\sum_{i=1}^d \partial_i (W_{,p_i}(x,u(x),\nabla u(x))}\varphi(x)\, dx$$

$$+ \underbrace{\int_{\partial\Omega} W_{,p}(x,u(x),\nabla u(x))\cdot n(x)\, \underbrace{\varphi(x)}_{=0 \text{ on } \partial\Omega}\, da}_{=0}$$

$$= \int_\Omega \Big(W_{,u}(x,u(x),\nabla u(x)) - \operatorname{div} W_{,p}(x,u(x),\nabla u(x)) \Big) \varphi(x)\, dx$$

W smooth, u smooth $\qquad \forall \varphi : \overline{\Omega} \to \mathbb{R}$, $\varphi|_{\partial\Omega} = 0$



Fundamental Lemma $\Longrightarrow$

$$(P) \qquad -\operatorname{div}\big(W_{,p}(x,u(x),\nabla u(x))\big) + W_{,u}(x,u(x),\nabla u(x)) = 0 \quad \forall x \in \Omega$$

partial differential equation (strong form)

We call $(\tilde{P})$ the weak form of (P)!

in the examples:

(i)' $\quad W_{,p}(x,u,p) = a\,p \qquad W_{,u}(x,u,p) = -f$

$$\Rightarrow (P) \qquad -\operatorname{div}\big(a(x)\nabla u(x)\big) = f(x) \quad \forall x \in \Omega$$

$\boxed{a = \mathbb{1}}$ $\qquad -\operatorname{div}(\nabla u(x)) = -\sum_{j=1}^{d} \partial_j \partial_j u(x) = \Delta u(x)$

$$(P) \quad \boxed{-\Delta u(x) = f(x)} \quad \forall x \in \Omega \quad \text{(Poisson – pb)}$$

(ii)

$$(P) \quad \boxed{-\operatorname{div}\left(\frac{\nabla u(x)}{\sqrt{1 + |\nabla u(x)|^2}} \right) = 0}$$

(minimal surface pb)

important

- So far we have assumed sufficient smoothness of a involved fcts

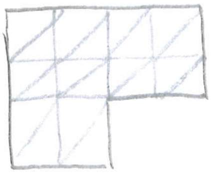
  pb: diffusivity $a : \overline{\Omega} \to \mathbb{R}^{d,d}$ may jump at material interfaces $\Rightarrow a \notin C^0(\overline{\Omega}, \mathbb{R}^{d,d})$

- We have yet not specified any function space!

Finite Elements at a glance:

Study $\mathcal{E}[\cdot]$ on some finite dimensional function space!

Ω polygonally bd. $\rightsquigarrow$ consider a triangulation $\mathcal{T}_h$ of Ω (more precise later!)



T triangle $\in \mathcal{T}_h$

h grid size

$$\mathcal{V}_h := \{ u_h : \Omega \to \mathbb{R} \mid u_h|_T \in \mathcal{P}^k, \, u \in C^0(\bar{\Omega}) \}$$

$\uparrow$ space of polynomials of degree $\leq k$

$$\mathcal{V}_{h,0} = \{ \varphi_h \in \mathcal{V}_h \mid \varphi_h|_{\partial\Omega} = 0 \}$$

(E_h) Minimize $\mathcal{E}[\cdot]$ over all functions $u_h \in \mathcal{V}_h$ with $u_h|_{\partial\Omega} = u_h^\partial$ ($u_h^\partial \in \mathcal{V}_h$ given approx. of u^∂ on $\partial\Omega$)

necessary condition:

$$(\tilde{P}_h) \quad 0 = \frac{d}{ds} \mathcal{E}[u_h + s\varphi_h] \Big|_{s=0} \quad \forall \varphi_h \in \mathcal{V}_{h,0}$$

$$= \int_\Omega W_{,u}(x, u_h(x), \nabla u_h(x)) \varphi_h(x) + W_{,p}(x, u_h(x), \nabla u_h(x)) \cdot \nabla \varphi_h(x) \, dx \quad \forall \varphi_h \in \mathcal{V}_{h,0}$$

(non) linear system of equations:

let $\{\varphi_h^i\}_{i \in I_h^0}$ Basis of $\mathcal{V}_{h,0}$ $\quad$ ($\{\varphi_h^i\}_{i \in I_h}$ Basis von $\mathcal{V}_h$)

$$F_h^i[u_h] := \int_\Omega W_{,u}(x, u_h, \nabla u_h) \varphi_h^i + W_{,p}(x, u_h, \nabla u_h) \nabla \varphi_h^i \, dx$$

$$(\tilde{P}_h) \quad \boxed{F_h[u_h] = \left(F_h^i[u_h] \right)_{i \in I_h^0} = 0}$$

with $u_h \in u_h^\partial + \mathcal{V}_{h,0}$ $\quad \left(u_h = u_h^\partial + \sum_{j \in I_h^0} U_h^j \varphi_h^j \right)$

$F_e : \mathbb{R}^{I_e^0} \to \mathbb{R}^{I_e^0}$ with

$$F_e[(u_e^j)_{j\in I_e^0}] := \mathcal{F}_e[u_e^\partial + \sum_{j\in I_e^0} u_e^j \varphi_e^j]$$

$\Rightarrow (\tilde{P}_e) \Leftrightarrow$ $\boxed{\text{Find root of } F_e \text{ on } \mathbb{R}^{I_e^0}}$

in the examples:

$u_e^\partial = \sum_{j\in I_e} u_e^{\partial, j} \varphi_e^j$

(i)' $$\underbrace{\int_\Omega a \sum_{j\in I_e^0} u_e^j \nabla\varphi_e^j \cdot \nabla\varphi_e^i \, dx}_{A_e \mathcal{U}_e} = \underbrace{-\int_\Omega a \nabla u_e^\partial \cdot \nabla\varphi_e^i \, dx + \int_\Omega f \varphi_e^i \, dx}_{R_e} \quad \forall_{i\in I_e^0}$$

with
$$\begin{aligned} A_{e,ij} &= \int_\Omega a \nabla\varphi_e^j \cdot \nabla\varphi_e^i \, dx \\ R_{e,i} &= -\int_\Omega a \nabla u_e^\partial \cdot \nabla\varphi_e^i \, dx + \int_\Omega f \varphi_e^i \, dx \\ \mathcal{U}_e &= (u_e^j)_{j\in I_e^0} \end{aligned}$$

$A_e \mathcal{U}_e = R_e$ linear system of equation

(A_e stiffness matrix)

reformulation:
$$a(u,v) = \int_\Omega a \nabla u \cdot \nabla v \, dx$$
$$\ell(v) = -a(u^\partial, v) + \int_\Omega f v \, dx$$

$(\tilde{P})$ Find $u^0 \in \mathcal{V}_0$ ($\mathcal{V}$ to be identified), s.t.

$a(u^0, \varphi) = \ell(\varphi) \quad \forall \varphi \in \mathcal{V}_0$

Interpretation: Find a representation u of the linear form $\ell(\cdot)$ in the quadratic form $a(\cdot,\cdot)$

$(\tilde{P}_e)$ Find $u_e^0 \in \mathcal{V}_{e,0}$, s.t.

$a(u_e^0, \varphi_e) = \ell_e(\varphi_e) \quad \forall \varphi_e \in \mathcal{V}_{e,0}$

where $l_h(\varphi_h) = -a(u_h^\partial, \varphi_h) + \int_\Omega f \varphi_h \, dx$

(ii)
$$\underbrace{\int_\Omega \frac{\sum_{j\in I_h} u_h^j \nabla \varphi_h^j}{\sqrt{1 + \left|\sum_{j\in I_h} u_h^j \nabla\varphi_h^j\right|^2}} \cdot \nabla \varphi_h^i \, dx}_{\int_\Omega \frac{\nabla u_h}{\sqrt{1+|\nabla u_h|^2}} \cdot \nabla\varphi_h^i \, dx} = 0 \quad \Leftrightarrow \quad F_h[U_h] = 0 \quad \forall i \in I_h^0$$

2nd variation:

$$\frac{d}{ds} \mathcal{F}^i[u_h + s\varphi_h^j]\Big|_{s=0} = \underbrace{\int_\Omega \frac{\nabla\varphi_h^j \cdot \nabla\varphi_h^i}{\sqrt{1+|\nabla u_h|^2}} - \frac{(\nabla u_h \cdot \nabla\varphi_h^i)(\nabla u_h \cdot \nabla \varphi_h^j)}{(1+|\nabla u_h|^2)^{\frac{3}{2}}} \, dx}_{=: A_{h,ij}[U_h]} \qquad i,j \in I_h^0$$

Newton Scheme:

$$\underbrace{A_h[U_h]}_{\in \mathbb{R}^{I_h^0, I_h^0}} \underbrace{(U_h^{m} - U_h^m)}_{\in \mathbb{R}^{I_h^0}} = -\underbrace{F_h[U_h^m]}_{\in \mathbb{R}^{I_h^0}}$$

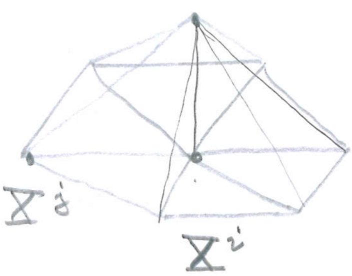
Simplest Finite Element ansatz:

$k = 1$ p.w. affine Finite Elements

degrees of freedom $\hat{=}$ nodal values at vertices of $\mathcal{T}_h$

basis fct's:

$\varphi_h^i(X^j) = \delta_{ij}$



hat - basis

$\rightsquigarrow$ A_h sparse $\quad A_{h,ij} = 0$ if X^j and X^i are not connected by an edge and $i \neq j$

outline of a rigorous theory (review):

- proper function spaces
- existence of weak solutions
- triangulations and FE spaces + adaptive meshes
- interpolation and error estimates

Literature: Ciarlet, The Finite Element Method for Elliptic Pbs '80
Braess, Finite Elemente '92
Brenner / Scott, The Mathematical Theory of Finite Element Methods '02

1. Existence of weak solution and Finite Element approximation

L^p spaces: $\Omega \subset \mathbb{R}^d$, open

$$L^p(\Omega) = \{ u : \Omega \to \mathbb{R} \mid u \text{ measurable}, \|u\|_p < \infty \}$$

with $\|u\|_{p<\infty} := \left(\int_\Omega |u|^p \, dx \right)^{\frac{1}{p}}$, $\|u\|_\infty := \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$

$(u,v)_2 := \int_\Omega u\, v \, dx$ scalar product on $L^2(\Omega)$

Hölder ineq. $\int_\Omega u\, v\, dx \leq \|u\|_p \|v\|_{p'}$

$\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p \leq \infty$ $\left(\frac{1}{\infty} = 0\right)$

$\|u+v\|_p \leq \|u\|_p + \|v\|_p$

classical fct spaces:

$$C^m(\bar\Omega) = \{ f : \bar\Omega \to \mathbb{R} \mid f \text{ m times cont. differentiable in } \Omega \text{ with continuously extensible derivatives on } \bar\Omega \}$$

$$\|f\|_{C^m(\bar\Omega)} = \sum_{|\beta| \leq m} \|\partial^\beta f\|_\infty \quad (\beta \text{ multiindex})$$

$$C^{m,\alpha}(\bar\Omega) \leftarrow C^m(\bar\Omega) \;\&\; \sup_{x,y} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} \leq C < \infty \quad |\beta| = m$$

weak derivatives:

$u \in L^p(\Omega)$ has a weak derivative $u^{(\beta)}$, if

$$\int_\Omega u\, \partial^\beta \varphi \, dx = (-1)^{|\beta|} \int_\Omega u^{(\beta)} \varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

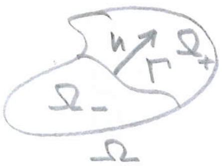
compare integration by parts, if u is smooth:

$$\int_\Omega u\, \partial_i \varphi \, dx = - \int_\Omega \partial_i u \, \varphi \, dx + \underset{\varphi|_{\partial\Omega}=0}{0}$$

iteratively

$$\int_\Omega u\, \partial^\beta \varphi \, dx = (-1)^{|\beta|} \int_\Omega \partial^\beta u \, \varphi \, dx$$

example:



u p.w. smooth

$$\int_\Omega u\, \partial_i \varphi \, dx = \int_{\Omega_- \cup \Omega_+} u\, \partial_i \varphi \, dx$$

$$= - \int_{\Omega_- \cup \Omega_+} \partial_i u \, \varphi \, dx + \int_\Gamma \underbrace{(u^- - u^+)}_{=0} n \, \varphi \, da$$

$\Rightarrow$ u is weakly differentiable

FE fcts are weakly differentiable

weak derivatives are unique!

Sobolev spaces

$1 \le p \le \infty, \quad m \in \mathbb{N}_0$

$$H^{m,p}(\Omega) := \{ u \in L^p(\Omega) \mid u \text{ has weak derivatives } u^{(\beta)} \in L^p(\Omega) \text{ for } 0 \le |\beta| \le m \}$$

$$\|u\|_{m,p} := \Big(\sum_{|\beta| \le m} \|u^{(\beta)}\|_p^p \Big)^{\frac{1}{p}} \qquad (u^{(0)} = u)$$

scalar product on $H^{m,2}(\Omega)$:

$$(u,v)_{m,2} := \sum_{|\beta| \le m} \int_\Omega u^{(\beta)} v^{(\beta)} \, dx$$

$H^{m,p}(\Omega)$ is a Banach space ($p=2$ Hilbertspace)

$$H^{m,p}(\Omega) = \overline{C^{\infty}(\Omega) \cap H^{m,p}(\Omega)}^{\|\cdot\|_{m,p}}$$

$$H_0^{m,p}(\Omega) = \overline{C_0^{\infty}(\Omega) \cap H^{m,p}(\Omega)}^{\|\cdot\|_{m,p}}$$

trace theorem $\Omega \subset \mathbb{R}^d$ open, Lipschitz bd.

$\exists\ B: H^{1,p}(\Omega) \to L^p(\partial\Omega)$ bounded, linear mapping with

$Bu = u$ on $\partial\Omega$ for $u \in H^{1,p}(\Omega) \cap C^0(\overline{\Omega})$

$$H_0^{1,p}(\Omega) = \{ u \in H^{1,p}(\Omega) \mid Bu = 0 \text{ on } \partial\Omega \} \subsetneq H^{1,p}(\Omega)$$

dual spaces $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$

$$H^{-m,p'}(\Omega) := \left(H_0^{m,p}(\Omega)\right)'$$

examples

- $f \in L^{p'}(\Omega)$ $\quad \langle f, u \rangle := \int_\Omega f u \, dx \Rightarrow$

  $|\langle f, u\rangle| \le \|f\|_{p'} \|u\|_p \Rightarrow \langle f, \cdot \rangle \in (L^p(\Omega))'$
  $\hat{=} L^{p'}(\Omega)$

  $\langle f, \cdot \rangle \in H^{m,p'} \quad \forall m \ge 0$

- $f \in L^{p'}(\partial\Omega)$ $\quad \langle f, u \rangle := \int_{\partial\Omega} f u \, da \overset{\text{trace theorem}}{\Rightarrow}$

  $\langle f, u \rangle \le \|f\|_{p',\partial\Omega} \|\underbrace{u}_{Bu}\|_{p,\partial\Omega} \le C \|f\|_{p',\partial\Omega} \|u\|_{1,p,\Omega}$

  $\Rightarrow \langle f, \cdot \rangle \in H^{-1,p'}(\Omega)$

Poincaré inequalities:

Ω open, bdd set, Lipschitz boundary, $d_\Omega = \mathrm{diam}(\Omega)$, then

(i) $\|u\|_{p,\Omega} \leq d_\Omega \|\nabla u\|_{p,\Omega} \quad \forall u \in H_0^{1,2}(\Omega) \quad \left(\|\nabla u\|_{p,\Omega} = \left(\sum \|\partial_i u\|_{p,\Omega}^p\right)^{\frac{1}{p}}\right)$

(ii) $\|u - \fint_\Omega u\|_{p,\Omega} \leq C_1 \|\nabla u\|_{p,\Omega} \quad \forall u \in H^{1,2}(\Omega)$

Proof for (i):

extend u by 0 outside Ω and assume without restriction $\Omega \subset [0, d_\Omega]^d$

$$u(x_1, x_{2-d}) = \int_0^{x_1} \partial_{x_1} u(\xi, x_{2-d})\, d\xi \implies$$

$$\int_\Omega |u|^p dx = \int_\Omega \left|\int_0^{x_1} 1 \cdot |\partial_{x_1} u(\xi, x_{2-d})|\, d\xi\right|^p dx$$

$$\underset{\text{Hölder-ineq.}}{\leq} \int_\Omega \int_0^{d_\Omega} |\partial_{x_1} u|^p d\xi\, d_\Omega^{\frac{p}{p'}}\, dx$$

$\frac{p}{p'} = \frac{p-1}{p} p = p-1$

$$= d_\Omega^{p-1} d_\Omega \int_\Omega |\partial_{x_1} u|^p d\xi\, dx_{2-d} = d_\Omega \|\partial_{x_1} u\|_{p,\Omega}^p$$

we already integrated over x_1

$\Rightarrow$ claim $\square$

Now we study existence of weak solution of linear elliptic PDEs ($(\tilde{P})$ for example (ii))

$\Leftrightarrow$ Find $u^\circ \in \mathcal{V}_0$, such that

$$a(u^\circ, \varphi) = \ell(\varphi) \quad \forall \varphi \in \mathcal{V}_0$$

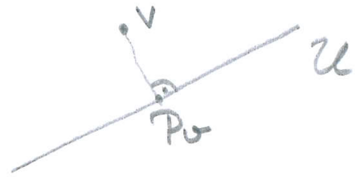
$(\tilde{P})$ see above

Choice of $\mathcal{V}_0$: $H_0^{1,2}(\Omega)$

Tools from functional analysis:

1.1. Thm (Projection theorem)

$\mathcal{U}$ closed subspace of a Hilbert space $\mathcal{V}$ and $v \in \mathcal{V}$, then there exist a unique $Pv \in \mathcal{U}$ with



$$\|v - Pv\| = \inf_{u \in \mathcal{U}} \|v - u\|$$

Furthermore P is a linear, bounded mapping $V \to \mathcal{U}$ and $v - Pv \perp \mathcal{U}$

1.2. Thm (Riesz representation theorem)

V Hilbert space, $\ell \in V'$ (bd, linear fct'nal), then there exists a unique $u \in V$, such that

$$(u, v) = \ell(v) \quad \forall v \in \mathcal{V}$$

Furthermore $\|u\|_{\mathcal{V}} = \|\ell\|_{\mathcal{V}'}$.

Proof:

$N = \operatorname{Ker} \ell$ closed subspace without restriction $N \neq \mathcal{V}$

$$\overset{1.1.}{\Longrightarrow} \exists\, u_0 \perp N \text{ with } \ell(u_0) \neq 0$$

$$\ell(\underbrace{v - \frac{\ell(v)}{\ell(u_0)} u_0}_{\in N}) = 0 \Rightarrow \left(u_0, v - \frac{\ell(v)}{\ell(u_0)} u_0\right) = 0 \Rightarrow$$

$$(u_0, v) = \frac{\ell(v)}{\ell(u_0)} \|u_0\|^2$$

$$u := \frac{\ell(u_0)}{\|u_0\|^2} u_0 \Longrightarrow (u, v) = \frac{\ell(u_0)}{\|u_0\|^2} \frac{\|u_0\|^2}{\ell(u_0)} \ell(v) = \ell(v)$$

$(u,u) = \ell(u) \Rightarrow \|u\|_V^2 \le \|\ell\|_{V'} \|u\| \Rightarrow \boxed{\|u\|_V \le \|\ell\|_{V'}}$

$$\|\ell\|_{V'} = \sup \frac{(u,v)}{\|v\|_V} \underset{\text{Cauchy-Schwarz}}{\le} \|u\|_V \frac{\|v\|_V}{\|v\|_V} \Rightarrow \boxed{\|\ell\|_{V'} \le \|u\|_V}$$

Uniqueness: $\left.\begin{aligned}(u,v) &= \ell(v)\\ (\tilde u, v) &= \ell(v)\end{aligned}\right\} \Rightarrow (u-\tilde u, v) = 0 \ \forall_v \Rightarrow u = \tilde u$ $\square$

1.3. Thm (Lax-Milgram)

V Hilbert space, $a: V \times V \to \mathbb{R}$ bilinear form with

$|a(v,w)| \le C \|v\| \|w\|$ (bounded), $a(v,v) \ge c \|v\|^2$ (coercive) $(0 < c \le C < \infty)$

then for each $\ell \in V'$ there exists a unique $u \in V$, such that

$$a(u,v) = \ell(v) \quad \forall v \in V$$

Furthermore $\|u\|_V \le \frac{1}{c} \|\ell\|_{V'}$.

Remark: $\boxed{a(u,v) = (u,v) \Rightarrow (1.3) \triangleq (1.2)}$

Proof (i) fix u and consider the fct'nal

$$v \longmapsto a(u,v)$$

linear ✓, bd ($a(u,v) \le C \|u\| \|v\|$)

$\overset{1.2.}{\Longrightarrow} \exists!\ w_u$ with $(w_u, v) = a(u,v)$

Now define $\boxed{T: V \to V;\ u \mapsto w_u}$

(ii) properties of T:

T linear: $(w_{u_1} + \alpha w_{u_2}, v) = a(u_1 + \alpha u_2, v)$

$= (w_{u_1 + \alpha u_2}, v) \quad \forall v \in V$

$\Rightarrow T(u_1 + \alpha u_2) = Tu_1 + \alpha Tu_2$

T injective: $Tu = 0 \Rightarrow 0 = (w_u, u) = a(u,u) \geq c\|u\|^2$
$\Rightarrow u = 0$

T bounded: $\|Tu\|^2 = (Tu, Tu) = a(u, Tu) \leq C_1 \|u\| \|Tu\|$
$\Rightarrow \|Tu\| \leq C_1 \|u\| \Rightarrow |||T||| \leq C_1$

Im(T) closed: $Tu_k \xrightarrow[k\to\infty]{} w \Rightarrow$

$$\|u_k - u_\ell\|^2 \leq \frac{1}{c} a(u_k - u_\ell, u_k - u_\ell)$$
$$\leq \frac{1}{c} (Tu_k - Tu_\ell, u_k - u_\ell)$$
$$\leq \frac{1}{c} \|Tu_k - Tu_\ell\| \|u_k - u_\ell\| \Rightarrow$$
$$\|u_k - u_\ell\| \leq \frac{1}{c} \|Tu_k - Tu_\ell\| \xrightarrow[k,\ell\to\infty]{} 0$$

$\Rightarrow (u_k)_k$ Cauchy sequence $\Rightarrow u_k \xrightarrow[k\to\infty]{} u \in V$

$\Rightarrow \|Tu_k - Tu\| \leq |||T||| \|u_k - u\| \longrightarrow 0 \Rightarrow Tu = w$

$\Rightarrow$ Im(T) closed

T surjective: assume T is not surjective $\Rightarrow \exists u \in V$ $u \notin Im(T)$
and $u \perp Im(T)$ ($\Leftarrow$ 1.1.)

$\Rightarrow \|u\|^2 \leq \frac{1}{c} a(u,u) = (Tu, u) = 0 \to u = 0$ ↯

altogether: $\boxed{\text{T is a bijective, bounded, linear mapping}}$

$\ell \in V' \overset{1.2.}{\Rightarrow} \exists w \in V \quad (w, v) = \ell(v) \quad \forall v \in V$

$u := T^{-1} w \Rightarrow a(u,v) = (Tu, v) = (w, v) = \ell(v) \quad \forall v \in V$

furthermore: $c\|u\|^2 \leq a(u,u) \leq \|\ell\|_{V'} \|u\| \Rightarrow \|u\| \leq \frac{1}{c} \|\ell\|_{V'}$

□

Weak formulation of Poisson's pb:

$$(P)\qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= u^\partial \quad \text{on } \partial\Omega \end{aligned}$$

weak form: $f \in L^2(\Omega)$, $u^\partial \in H^{1,2}(\Omega)$ $(u = u^0 + u^\partial)$

$$(\tilde{P})\qquad \underbrace{\int_\Omega \nabla u^0 \cdot \nabla \varphi}_{a(u^0,\varphi)} = \underbrace{\int_\Omega f\varphi - \nabla u^\partial \cdot \nabla \varphi \, dx}_{\ell(\varphi)}$$

natural space $V_0 = H_0^{1,2}(\Omega)$

$\boxed{\text{to apply Riesz representation thm:}}$

(i) $a(\cdot,\cdot)$ is a scalar product on $H_0^{1,2}(\Omega)$, which is equivalent to $(\cdot,\cdot)_{1,2}$:

$$(u,u)_{1,2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2$$

$$\underset{\text{Poincaré}}{\leq} (1 + d_\Omega) \|\nabla u\|_2^2 = (1 + d_\Omega)\, a(u,u)$$

$$a(u,u) \leq (u,u)_{1,2}^2$$

$\Rightarrow$ in particular norms $\|u\|_{1,2}$ and $\sqrt{a(u,u)}$ are equivalent.

$$\left.\begin{aligned} &a(\cdot,\cdot) \text{ symmetric, bilinear} \\ &a(u,u) = 0 \Rightarrow \|u\|_{1,2} = 0 \Rightarrow u = 0 \end{aligned}\right\} a(\cdot,\cdot) \text{ scalar product}$$

(ii) $\ell(\cdot)$ is a bounded linear functional on $H_0^{1,2}(\Omega)$:

$$\ell(\varphi) = \int_\Omega f\varphi - \nabla u^\partial \cdot \nabla \varphi \, dx \leq \|f\|_2 \|\varphi\|_2 + \|\nabla u^\partial\|_2 \|\nabla \varphi\|_2$$

$$\leq \|f\|_2 \|\varphi\|_{1,2} + \|u^\partial\|_{1,2} \|\varphi\|_{1,2} \leq (\|f\|_2 + \|u^\partial\|_{1,2}) \|\varphi\|_{1,2}$$

1.2.
$\Rightarrow$ $\quad \exists\, u^0 \in H_0^{1,2}(\Omega) \quad a(u^0,\varphi) = \ell(\varphi) \Leftrightarrow (\tilde{P})$

$V = V_0 = H_0^{1,2}(\Omega)$

$$\underset{u = u^0 + u^\partial}{\Leftrightarrow} \boxed{\int_\Omega \nabla u \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx \quad \forall \varphi \in H_0^{1,2}(\Omega)}$$

Alternative: [directly with Lax-Milgram based on (i), (ii)

$$\left((i) \Rightarrow \quad \begin{array}{l} a(u,v) \le \|u\|_{1,2} \|v\|_{1,2} \\ a(u,u) \ge \dfrac{1}{1+d_\Omega} \|u\|_{1,2}^2 \end{array} \right)$$
]

Lax-Milgram allows to study more general problems:

$$(Lu)(x) = -\underbrace{\sum_{i,j=1}^{d} \partial_i \big(a_{ij}(x)\, \partial_j u(x)\big)}_{= \operatorname{div}(a(x) \nabla u)} - \sum_{j=1}^{d} \partial_j \big(a_j(x) u(x)\big) + \sum_{j=1}^{d} b_j(x)\, \partial_j u(x) + a_0(x)\, u(x)$$

$\uparrow$ differential operator

with $a_{ij}, a_j, b_j, c_0 \in L^\infty(\Omega) \quad (i,j = 1, \dots d)$

if u, a_{ij}, a_j are differentiable, then

$$Lu = \sum_{i,j}^{d} a_{ij}\, \partial_i \partial_j u + \sum_{i,j}^{d} \tilde{b}_j\, \partial_j u + \tilde{a}_0 u$$

with $\tilde{b}_j = b_j - a_j - \sum_{i=1}^{d} \partial_i a_{ij}, \quad \tilde{a}_0 = a_0 - \sum_{j=1}^{d} \partial_j a_j$

Now we ask for a solution of the pb:

(P_L) Find $u : \overline{\Omega} \to \mathbb{R}$, such that

$$Lu = f \quad \text{in } \Omega$$
$$u = u^\partial \quad \text{on } \partial\Omega$$

weak formulation:

$(\tilde{P}_L)$ For $f \in L^2(\Omega)$, $u^\partial \in H^{1,2}(\Omega)$, find $u = u^0 + u^\partial$ with $u^0 \in H_0^{1,2}(\Omega)$, such that

$$\underbrace{\int_\Omega \sum_{i,j}^d a_{ij}\, \partial_j u\, \partial_i \varphi + \sum_{j=1}^d a_j\, u\, \partial_j \varphi + \sum_{j=1}^d b_j\, \partial_j u\, \varphi + a_0 u \, dx}_{=:\, a(u,\varphi)}$$

$$= \int_\Omega f \varphi \, dx \qquad \forall \varphi \in H_0^{1,2}(\Omega)$$

$$\Leftrightarrow \quad a(u,\varphi) = \int_\Omega f\varphi\, dx \quad \forall \varphi \in H_0^{1,2}(\Omega)$$

$$\Leftrightarrow \quad a(u^0,\varphi) = \int_\Omega f\varphi\, dx - a(u^\partial,\varphi) \quad \forall \varphi \in H_0^{1,2}(\Omega)$$

1.4. Def (uniform elliptic, L-condition)

(i) $L_0 u := \sum_{i,j}^d \partial_i (a_{ij} \partial_j u)$ is uniformly elliptic if $\sum_{i,j=1}^d a_{ij} \xi_i \xi_j \geq c_0 |\xi|^2 \quad \forall \xi \in \mathbb{R}^d$ with $c_0 > 0$

(ii) L fulfills the L-condition (Ladyzhenskaya) if

$$\sum_{i,j=1}^d a_{ij} \xi_i \xi_j + \sum_{j=1}^d (a_j + b_j)\xi_j \alpha + a_0 \alpha^2 \geq c_0 |\xi|^2$$

$\forall \xi \in \mathbb{R}^d$, $\alpha \in \mathbb{R}$ with $c_0 > 0$

($\alpha = 0 \Rightarrow$ uniform ellipticity)

1.5 Thm (existence of weak solutions)

Under the conditions

$f \in L^2$, $u^\partial \in H^{1,2}$, $a_{ij}, a_j, b_j, a_0 \in L^\infty$

and the L-condition

there exists a unique solution u of $(\tilde{P}_L)$ in $H^{1,2}(\Omega)$

and $\|u\|_{1,2} \leq C_1 \left(\|f\|_2 + \|u^\partial\|_{1,2}\right)$

Remark [$a(\cdot,\cdot)$ in general not symmetric
$\to$ no scalar product $\to$ no direct application of 1.2.]

Proof

(i) $$|a(v,w)| \le C(\max_{i,j}\|a_{ij}\|_\infty)\,\|\nabla v\|_2\|\nabla w\|_2$$
$$+ C(\max_j\|a_j\|_\infty)\|\nabla w\|_2\|v\|_2 + C(\max_j\|b_j\|_\infty)\|\nabla v\|_2\|w\|_2$$
$$+ \|a_0\|_\infty\|v\|_2\|w\|_2$$
$$\le C\,\underbrace{(\|v\|_2+\|\nabla v\|_2)}\,(\|w\|_2+\|\nabla w\|_2)$$
$$\le C\,\|v\|_{1,2}\,\|w\|_{1,2}$$

[$\sqrt{a^2+b^2}$, $|a|+|b|$ are equivalent norms on $\mathbb{R}^2$]

(ii) $$a(v,v) \underset{\substack{\uparrow\\ \text{L-condition}}}{\ge} c_0\|\nabla v\|_2^2 \underset{\substack{\uparrow\\ \text{Poincaré}}}{\ge} c_0\left(\frac{\|\nabla v\|_2^2}{2}+\frac{\|v\|_2^2}{2d_\Omega}\right)$$
$$= \frac{c_0}{2}\left(1+\frac{1}{d_\Omega}\right)\|v\|_{1,2}^2$$

(iii) (i) & assumption $\Rightarrow$
$$\ell(v) \le C\|u^\partial\|_{1,2}\|v\|_{1,2} + \|f\|_2\|v\|_2$$
$$\le C\,\|v\|_{1,2}$$

$\overset{1.3.}{\Longrightarrow}$ $\exists!\, u^0\in H_0^{1,2}(\Omega)$ $\quad a(u^0,\varphi) = (f,\varphi) - a(u^\partial,\varphi)$

$\Rightarrow$ $u = u^0+u^\partial$ is the unique solution of $(\tilde{P}_L)$

$$|\ell(v)| \le C(\|u^\partial\|_{1,2}+\|f\|_2) \overset{1.3.}{\Longrightarrow}$$
$$\|u^0\|_{1,2} \le C(\|u^\partial\|_{1,2}+\|f\|_2) \Rightarrow$$

$$\|u\|_{1,2} \le \|u^0\|_{1,2} + \|u^\partial\|_{1,2} \le C'\left(\|u^\partial\|_{1,2} + \|f\|_2\right)$$

□

1.6 Example (nonsmooth solutions)

1D "$-\mathrm{div}(a\nabla u) = 0$" $\quad \Omega = (0,1)$, $f \equiv 0$, $u^\partial(0) = 0, u^\partial(1) =$

weak notion:
$$\int_0^1 a (u^0)' \varphi' \, dx = -\int_0^1 a \underbrace{(u^\partial)'}_{=1 \ (u^\partial(x) := x)} \varphi' \, dx$$

$$a = \begin{cases} a_0 & ; x \le \frac{1}{2} \\ a_1 & ; x > \frac{1}{2} \end{cases} \qquad a_0 = 1, \quad a_1 = 2$$

For supp $\varphi \subset (0, \frac{1}{2})$:

$$\int_0^1 a_0 (u^0)' \varphi' \, dx = -\int_0^1 a_0 \varphi' \, dx \quad \xRightarrow{\text{integration by parts}}$$

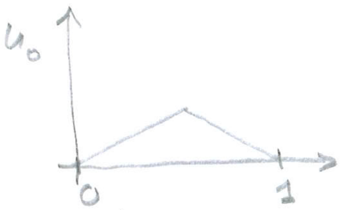
$$\int_0^1 (a_0 (u^0)')' \varphi \, dx + 0 = +0 - 0 \quad \xRightarrow{\forall \varphi, \ \mathrm{supp}\,\varphi \subset (0,\frac{1}{2})} \quad \underbrace{(a_0 (u^0)')'(x)}_{a_0 (u^0)''(x)} = 0 \ \forall x \in (0, \tfrac{1}{2})$$

no bd terms

$$\Rightarrow (u^0)''(x) = 0 \Rightarrow \boxed{u^0 \text{ affine on } (0, \tfrac{1}{2})}$$

analogously: $\boxed{u^0 \text{ affine on } (\tfrac{1}{2}, 1)}$

$\boxed{\text{Ansatz}}$: slope of u^0: s_0 on $(0, \frac{1}{2})$, s_1 on $(\frac{1}{2}, 1)$



$$u^0(0) = u^0(1) = 0 \Rightarrow s_0 + s_1 = 0$$

$$\underbrace{\int_0^1 a (u^0)' \varphi' \, dx}_{} = -\int_0^1 a \varphi' \, dx \quad \xRightarrow{\text{integration by parts}} \quad 0 + 0 - (a_0 - a_1)\varphi(\tfrac{1}{2})$$

$$= 0 + 0 + (a_0 s_0 - a_1 s_1)\varphi(\tfrac{1}{2})$$

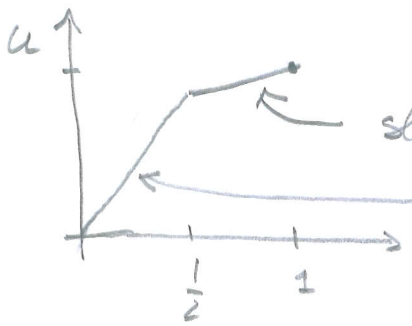
$$\Rightarrow \boxed{\text{linear system}} \qquad \begin{aligned} a_0 s_0 - a_1 s_1 &= a_1 - a_0 \\ s_0 + s_1 &= 0 \end{aligned}$$

$\Rightarrow \quad (a_0 + a_1) s_0 = a_1 - a_0$

$\boxed{s_1 = -s_0}$

$\Rightarrow \quad s_0 = \dfrac{a_1 - a_0}{a_1 + a_0}\,, \quad s_1 = \dfrac{a_0 - a_1}{a_1 + a_0}$

$\Rightarrow$



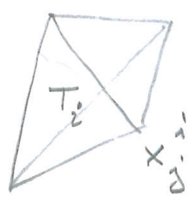
slope $1 + s_1 = \dfrac{2a_0}{a_1 + a_0} = \dfrac{2}{3}$

slope $1 + s_0 = \dfrac{2a_1}{a_1 + a_0} = \dfrac{4}{3}$

Now we study the Finite Element approximation:

$\boxed{\text{Triangulations}} \qquad \mathcal{T}_h = \{ T_i \mid i = 1, \ldots, m \}$,

regular simplex: $\quad T_i = \{ x = \lambda_0 x_0^i + \lambda_1 x_1^i + \ldots \lambda_d x_d^i \mid \lambda_i \geq 0,\ \sum_{i=0}^{d} \lambda_i = 1 \}$



simplex with vertices $x_0^i, \ldots x_d^i$

regular $\hat{=} \quad \det \begin{pmatrix} | & & | \\ x_1^i - x_0^i & \ldots & x_d^i - x_0^i \\ | & & | \end{pmatrix} \neq 0$

$T_i \cap T_j$ either empty or a common subsimplex (face, edge, vertex)

$\bar{\Omega} = \bigcup_{i=1}^{m} T_i$

$\mathcal{T}_h$ admissible triangulation

$h = \max_{i=1,\ldots m} \underbrace{\max_{k,j=0,\ldots d} |x_k^i - x_j^i|}_{h_T}$ (maximal diameter)

grid size as a function: $\quad h : \bar{\Omega} \to \mathbb{R}$

$h(x) = \max_{x \in T} h_T$

$$\varrho(T) = 2 \sup\{r \mid \exists x \in T \ B_r(x) \subset T\}$$
(diameter of the maximal inscribed ball)

$\{\mathcal{T}_h\}_h$ family of triangulation is regular, if

$$\sigma(\mathcal{T}_h) = \max_{T \in \mathcal{T}_h} \frac{h(T)}{\varrho(T)} \leq C_1 \quad (\ast)$$

So far: $\boxed{\text{Lagrangian Finite Elements } (T, P, \Gamma)}$

T — simplex

$P = \mathcal{P}_k$ — local, discrete fct'space: polynomials of degree $\leq k$

Lagrangian basis with nodes $\{q^\alpha\}_{|\alpha| \leq k}$, $\alpha \in \mathbb{N}_0^d$

cardinality $= \binom{d+k}{k}$

$\Gamma = \{\gamma^\alpha\}_{|\alpha| \leq k}$ — degrees of freedom (dof)

as linear fct'nals on C^∞ with

$\gamma^\alpha(\varphi) = \varphi(q^\alpha)$

More general:

1.7. Definition (Finite Element)

A triplet (T, P, Γ) is called finite element (in $\mathbb{R}^d$) if

- T is a non deg. d-Simplex
- P is a real-valued fct space on T — P finite dim.
- $\Gamma = \{\gamma^\alpha\}_{\alpha \in I_\Gamma}$ (index set) with

$\gamma^\alpha : C^\infty(T) \to \mathbb{R}$ with the property

$$\forall \alpha \in I_\Gamma \ \exists! \ p_\alpha \in P \quad \gamma^\beta(p_\alpha) = \delta_{\alpha\beta}$$

$\{p_\alpha\}_{\alpha \in I_\Gamma}$ is denoted local finite element basis

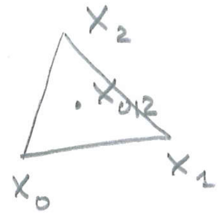
examples

- **cubic Hermit finite element**

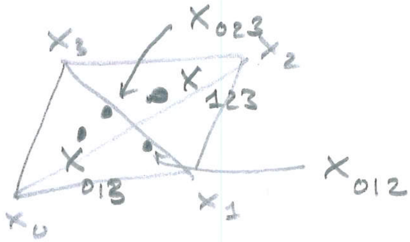
$\boxed{d=1}$ $T = [a,b]$, $P = \mathcal{P}_3$,

$\gamma^0(p) = p(a)$, $\gamma^1(p) = p'(a)$, $\gamma^2(p) = p(b)$, $\gamma^3(p) = p'(b)$

$\boxed{d=2}$



$\boxed{d=3}$



$x_{ijk} = \frac{x_i + x_j + x_k}{3}$

$$\Gamma = \Big\{ p(x_i)_{i=0,\dots,d},\ \nabla p(x_i)(x_j - x_i)_{\substack{i \neq j \\ i,j=0,\dots,d}},\ p(x_{ijk})_{\substack{i<j<k \\ i,j,k=0,\dots}} \Big\}$$

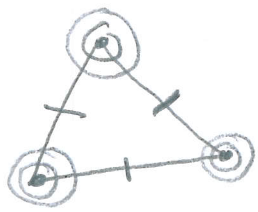
$\#\Gamma = 3 + 6 + 1 = 10 = \binom{2+3}{2}$ for $d=2$

$= 4 + 4\cdot 3 + 4 = 20 = \binom{3+3}{3}$ for $d=3$

remark: $\boxed{d=1}$ $\leadsto$ global C^1 fct's

$\boxed{d=2,3}$ missing C^1 smoothness in normal direction at simplex faces

- **Argyris-Element**



$\boxed{d=2}$ $(T, \mathcal{P}_5, \Gamma)$

$$\Gamma = \Big\{ p(x_i),\ \partial_j p(x_i),\ \partial_j \partial_k p(x_i),\ \partial_n p\big(\underbrace{\tfrac{x_i + x_e}{2}}_{x_{ie}}\big) \Big\}$$

$i = 0,1,2$
$j,k = 1,2$
$e = 0,1,2$
$e \neq i$

$\#\Gamma = 3 + 6 + 9 + 3 = 21 = \binom{5+2}{2}$

$\partial_n p\big|_{\substack{\text{edge} \\ x_i x_e}} \in \mathcal{P}_4 \iff$ FE fct's are C^1

$\partial_n p(x_i), \partial_n p(x_e), \partial_n p(x_{ie})$,
$\nabla(\partial_n p(x_i))(x_e - x_i), \nabla(\partial_n p(x_e))(x_i - x_e)$
prescribed

1.8. Definition (affine equivalence)

A finite element (T, P, Γ) is affine equivalent to an element $(\hat{T}, \hat{P}, \hat{\Gamma})$, if there is a bijective mapping

$F : \hat{x} \longmapsto A\hat{x} + b$ with

$F(\hat{T}) = T$, $P = \{ p : T \to \mathbb{R} \mid p = \hat{p} \circ F^{-1},\ p \in \hat{P} \}$,

for $\gamma \in \Gamma$ there exist a unique $\hat{\gamma} \in \hat{\Gamma}$ with $\gamma(f) = \hat{\gamma}(f \circ F)$

[concrete: $\gamma : p \longmapsto D^{s_\gamma} p(x^\gamma)(\xi^\gamma_1, \ldots \xi^\gamma_{s_\gamma})$

$\Rightarrow \hat{x}^\gamma = F(x^\gamma)$, $s^\gamma = s^{\hat{\gamma}}$, $\xi^\gamma_1 = A \hat{\xi}^{\hat{\gamma}}_{s_\gamma}$ (chain rule)]

1.9. Remark

- Lagrange-Element of kth order on regular Simplices are affine equivalent
- Argyris-Elements on $T = F(\hat{T})$, $\hat{T}$ are affine equivalent, iff $A \in SO(2)$

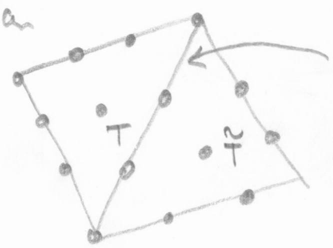
1.10 Definition (finite element space)

$\mathcal{T}_h$ regular triangulation, $\{(T, P_T, \Gamma_T)\}_{T \in \mathcal{T}_h}$ a family of finite element, which are equivalent to a (reference-) finite element $(\hat{T}, \hat{P}, \hat{\Gamma})$, then the associated finite element space is given as

$$V_h := \Big\{ v = (v_T)_{T \in \mathcal{T}_h} \in \prod_{T \in \mathcal{T}_h} P_T \ \Big|\ \gamma_T = \gamma_{\tilde{T}} \Rightarrow \gamma_T(v_T) = \gamma_{\tilde{T}}(v_{\tilde{T}}) \Big\}$$

example:

Lagrangian FE



$k = 3$

$\underset{\substack{\Uparrow \\ P}}{v_T}\big|_{T \cap \tilde{T}} = \underset{\substack{\Uparrow \\ \tilde{P}}}{v_{\tilde{T}}}\big|_{T \cap \tilde{T}} \in \mathcal{P}_3$ on the edge

1.11. **Definition** (interpolation in the local FE space)

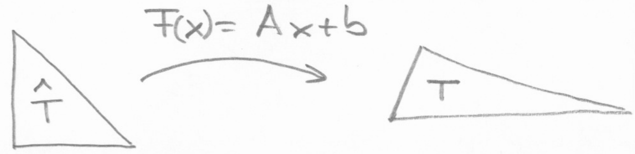
(T,P,Γ) finite element with basis $\{\varphi_\alpha\}_{\alpha\in I_\Gamma}$, the define the interpolation

$$\mathcal{I} : C^\infty(T) \to P \; ; \; v \mapsto \sum_{\alpha\in I_\Gamma} \gamma^\alpha(v)\,\varphi_\alpha$$

on P.

1.12. **Remark** For affine equiv. FEs (T,P,Γ), $(\hat T,\hat P,\hat\Gamma)$ we have $\widehat{\mathcal{I} v} = \hat{\mathcal{I}}\hat v$ with $\hat w = w\circ F$

Pf



$$\widehat{\mathcal{I} v} = \sum_{\alpha\in I_\Gamma} \gamma_\alpha(v)(\varphi_\alpha\circ F) = \sum_{\alpha\in I_\Gamma} \underbrace{\hat\gamma_\alpha(\hat v)}_{\text{(concrete:) chain rule}}\,\hat\varphi_\alpha = \hat{\mathcal{I}}\hat v$$

$$\boxed{\hat\gamma^\beta(\varphi_\alpha\circ F) \underset{\text{chain rule*}}{=} \gamma^\beta(\varphi_\alpha) = \delta_{\alpha\beta}}$$

(concrete) theory*: $$D^{s_\gamma}\hat v(\hat x^\gamma)(\xi_1^\gamma, \ldots, \xi_{s_\gamma}^\gamma) \underset{D^2F=0}{=} D^{s_\gamma}\underbrace{\hat v(\underbrace{F(\hat x^\gamma)}_{=x^\gamma})}_{=v(x^\gamma)}\underbrace{(A\xi_1^\gamma, \ldots, A\xi_{s_\gamma}^\gamma)}_{(\xi_1^\gamma, \ldots, \xi_{s_\gamma}^\gamma)}$$

□

1.13 **Definition** (interpolation on V_h)

$\mathcal{T}_h$ an admissible triangulation, V_h a finite element space, then define the interpolation

$$I_h : C^\infty(\overline{\Omega}) \to V_h \; ; \; I_h v|_T = (\mathcal{I}(v|_T))_{T\in\mathcal{T}_h}$$

Remark Up to a possible nonuniqueness on simplicial faces $v\in V_h$ can be considered as a function ($\in L^\infty$)

recall: (discrete) variational pb

$(\tilde{P}_L)$ find $u = \underset{\substack{\uparrow \\ H^{1,2}_0(\Omega)}}{u^0} + \underset{\substack{\uparrow \\ H^{1,2}(\Omega)}}{u^\partial}$ such that

$$a(u,\varphi) = (f,\varphi) := \int_\Omega f\varphi\,dx$$

$$\Leftrightarrow \quad a(u^0,\varphi) = \ell(\varphi) := (f,\varphi) - a(u^\partial,\varphi) \quad \forall \varphi \in H^{1,2}_0(\Omega)$$

finite element problem as a linear system:

$(\tilde{P}_{L,h})$ Find $u_h = u_h^0 + u_h^\partial$ with $u_h^\partial \in \mathcal{V}_h$ (given) and $u_h^0 \in \mathcal{V}_{h,0} := \{ v_h \in \mathcal{V}_h \mid v_h|_{\partial\Omega} = 0 \}$, such that

$$a(u_h,\varphi_h) = \ell_h(\varphi_h) := (f,\varphi_h) - a(u_h^\partial,\varphi_h)$$

$$\forall \varphi_h \in \mathcal{V}_h^0$$

$\Leftrightarrow \quad A_h X_h = R_h$ $\quad$ $m \times m$ linear system, $m = \dim \mathcal{V}_h^0$

with $(A_h)_{ij} = a(\varphi_h^j, \varphi_h^i)$ stiffness matrix

$(R_h)_i = \ell(\varphi_h^i)$

$(X_h)_i = \gamma_{\alpha(i)}(u_h)$ $\quad$ $\alpha(i)$: ith element in the global enumeration of all dofs of $\mathcal{V}_h^0$

How to assemble this system?

index mapping: $M(T,\alpha)$ global index associated to the dof γ^α on $T \in \mathcal{T}_h$

local stiffness matrix:

$(A^T_{h,\alpha\beta})_{\alpha,\beta \in I_{\Gamma_T}}$ defined as A_h but with integration only on T

$$\left(R^{T,f}_{e,\alpha}\right)_{\alpha\in I_{\Gamma_T}} = \int_T f\,\varphi_\alpha\,dx$$

assembly of global stiffness matrix and right hand side:

$A^{ext}_e = 0$; $R^{ext}_e = 0$; $\qquad A^{ext}_e \in \mathbb{R}^{\dim V_e,\,\dim V_e}$, $\quad R^{ext}_e \in \mathbb{R}^{\dim V_e}$

for all $T \in \mathcal{J}_e$ {
  for all $\alpha \in I_{\Gamma_T}$ {
    for all $\beta \in I_{\Gamma_T}$ {
      compute $A^T_{e,\alpha\beta}$; $A^{ext}_{e,M(T,\alpha)M(T,\beta)} \mathrel{+}= A^T_{e,\alpha\beta}$;
    }
    compute $R^{T,f}_{e,\alpha}$; $R^{ext,f}_{e,M(T,\alpha)} \mathrel{+}= R^T_{e,\alpha}$;
  }
}

probably in separate loops

$$R^{ext}_e = R^{ext,f}_e - A^{ext}_e X^\partial_e \quad \text{where } X^\partial_e = \left(\gamma_{\alpha(i)}(u^\partial_e)\right)_{i=1,\dots,\dim V_e}$$

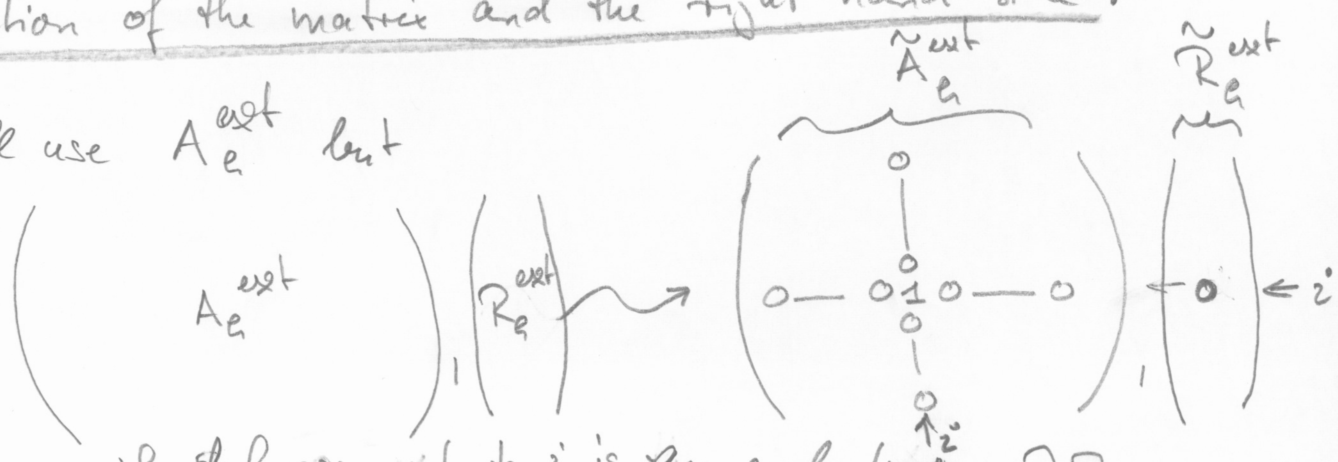
practical implementation ($\boxed{\text{eigen library}}$):

compute $A^T_{e,\alpha\beta}$ $\rightsquigarrow$ store in a list as an entry $(M(T,\alpha), M(T,\beta), A^T_{e,\alpha\beta})$

then perform global assembly based on this list (same with $R^T_{e,\alpha}$)

reduction of the matrix and the right hand side:

still use A^{ext}_e but



if dof associate to i is an evaluation on $\partial\Omega$

then solve

$$\tilde{A}_e^{ext} \underset{\underset{\mathbb{R}^{\dim V_e}}{\cap}}{\tilde{X}_e^{ext}} = \tilde{R}_e^{ext} \qquad \big(\text{i.e. } \underset{\uparrow \text{ bd. dof}}{X_{e,i}^{ext}} = 0 \big)$$

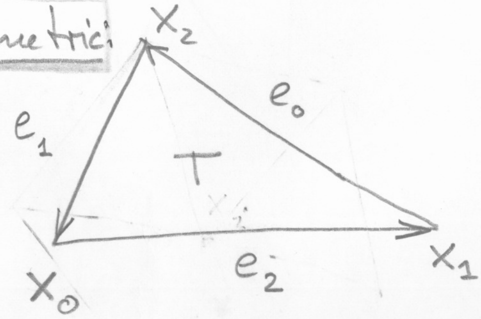
$$X_e^{ext} = \tilde{X}_e^{ext} + X_e^{\partial}$$

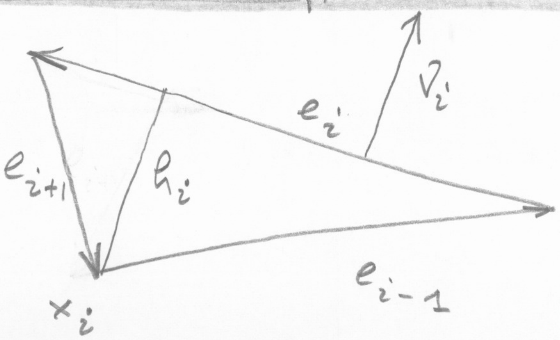
$$u_e = \sum_{i=1,\dots,\dim V_e} X_{e,i}^{ext} \varphi_{\alpha(i)}$$

Different ways to assemble the (local) stiffness matrix:

(1) geometric

$\boxed{\text{affine FE}}$





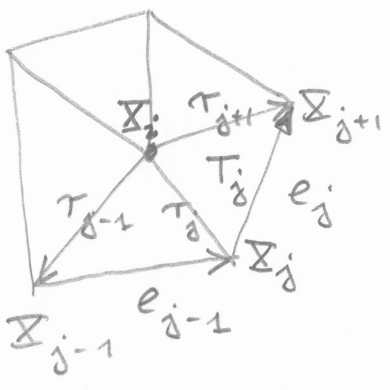
$$\nabla p_i = - \frac{\nu_i}{h_i}$$

$$\Rightarrow \quad A^T_{e,ij} = |T| \frac{\nu_i \cdot \nu_j}{h_i\, h_j}$$

$$= |T| \frac{e_i}{|e_i|} \cdot \frac{e_j}{|e_j|} \frac{1}{h_i\, h_j}$$

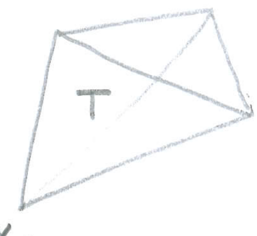
$$\boxed{2|T| = h_i |e_i|} \longrightarrow \quad = \frac{e_i \cdot e_j}{4\,|T|}$$

$\rightsquigarrow$ global stiffness matrix:



$$A_{e,ij} = \begin{cases} \displaystyle\sum_k \frac{|e_k|^2}{4|T_k|} & ; \ i = j \\[2ex] \displaystyle\frac{e_{j-1} \cdot \tau_{j-1}}{4\,|T_{j-1}|} - \frac{e_j \cdot \tau_{j+1}}{4\,|T_j|} & ; \ X_j \in adj(X_i) \end{cases}$$

2 affin FE (via barycentric coordinates):



$$\lambda_i = \varphi_i \quad i\text{th basis fct}$$

x_i

$$A \underbrace{\begin{pmatrix} \lambda_1(x) \\ | \\ \lambda_d(x) \end{pmatrix}}_{\lambda_{1-d}(x)} = x - x_0 \quad , \quad A = \begin{pmatrix} | & & | \\ x_1 - x_0 & - & x_d - x_0 \\ | & & | \end{pmatrix}$$

$$\Rightarrow \quad \lambda_{1-d}(x) = A^{-1}(x - x_0)$$

$$\Rightarrow \quad D\lambda_{1-d}(x) = A^{-1} \in \mathbb{R}^{d,d}$$

$\lambda_0 = 1 - \sum_{j=1}^{d} \lambda_j$

$$\Rightarrow \quad D\lambda_{0-d}(x) = \begin{pmatrix} A^{-1} \\ \sum_{i=1}^{d} (A^{-1})_{i} \end{pmatrix} \in \mathbb{R}^{d+1,d}$$

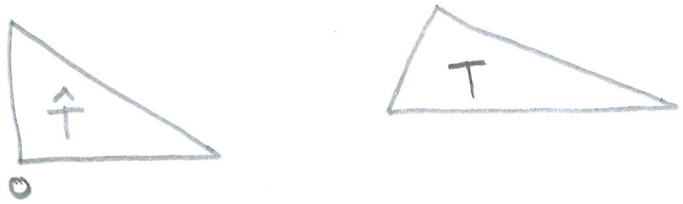
$\hat{i}$ rows index

$$\boxed{(P) \quad -\operatorname{div}(a \nabla u) = f}$$

$$\to \quad \mathbb{R}^{d+1,d+1} \ni A_{e}^{T} = \frac{\det A}{2} \left(a \nabla \lambda_i \cdot \nabla \lambda_j \right)_{i,j=0,-d} = \frac{\det A}{2} \left(\nabla \lambda_j^T a \nabla \lambda_i \right)_{ij=0,-d}$$

$$= \frac{\det A}{2} D\lambda_{0-d} \, a \, (D\lambda_{0-d})^T$$

3 general case (affin equivalent to a reference element):



$$\hat{u}(\hat{x}) = u(x)$$

$$x = A\hat{x} + x_0$$

$$A = D_{\hat{x}} x$$

$$A^T_{q,\alpha\beta} = \int_T a \, \nabla_x p_\alpha \cdot \nabla_x p_\beta \, dx$$

$$\nabla_x p_\alpha = D_x p_\alpha^T = \left(D_{\hat{x}} \hat{p}_\alpha \, D_x \hat{x}\right)^T = D_x \hat{x}^T \, \nabla_{\hat{x}} \hat{p}_\alpha$$

$$\Rightarrow \quad A^T_{q,\alpha\beta} = \int_{\hat{T}} (\nabla_{\hat{x}} \hat{p}_\beta)^T \left(D_x \hat{x} \; a \; D_x \hat{x}^T\right) \nabla_{\hat{x}} \hat{p}_\alpha \, \det D_{\hat{x}} x \, d\hat{x}$$

$$\approx \sum_{k=1}^{K} \hat{\omega}_k \underbrace{\left(\nabla_{\hat{x}} \hat{p}_\beta(\hat{y}_k)\right)^T}_{\text{can be precomputed}} \underbrace{A^{-1} a(y_k) A^{-T}}_{\text{geometry dependent}} \underbrace{\nabla_{\hat{x}} \hat{p}_\alpha(\hat{y}_k)}_{} \det A$$

Quadrature weights $\hat{\omega}_k$
nodes $\hat{y}_k$

---

in many case (e.g. Lagrangian FE):

$$\hat{p}_\beta(\hat{x}) = \tilde{p}_\beta(\lambda_0(\hat{x}), \ldots, \lambda_d(\hat{x}))$$

$$\lambda_1(\hat{x}) = \hat{x}_1 \qquad \lambda_0(\hat{x}) = 1 - \hat{x}_1 - \ldots - \hat{x}_d$$

$$\vdots$$

$$\lambda_d(\hat{x}) = \hat{x}_d$$

$$\Rightarrow \quad \nabla_{\hat{x}} \hat{p}_\beta = \underbrace{D_{\hat{x}} \lambda}_{\begin{pmatrix} -1 & \cdots & -1 \\ 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}} \, D_\lambda \tilde{p}_\beta$$

example: $\boxed{d=2}$ $\hat{p}_\beta(\hat{x}) = 4 \lambda_0(x) \lambda_1(x)$ (quadratic basis fct)

$$= 4(1 - \hat{x}_1 - \hat{x}_2)\, \hat{x}_1 \qquad \nabla_{\hat{x}} \hat{p}_\beta(\hat{x}) = \begin{pmatrix} -8\hat{x}_1 - 4\hat{x}_2 \\ -4\hat{x}_1 \end{pmatrix}$$

1.14 Thm (interpolation error estimate)

$\mathcal{T}_h$ regular, admissible triangulation, V_h FE space with elements (T, P, Γ) affine equivalent to a reference element $(\hat{T}, \hat{P}, \hat{\Gamma})$, $V_h \subset H^{m,p}(\Omega)$, $\mathcal{P}_k \subseteq \hat{P}$, $u \in H^{k+1,p}(\Omega)$, $H^{k+1,p}(\hat{\Omega}) \overset{*}{\hookrightarrow} C^{\max s_{\hat{\gamma}}}(\overline{\Omega})$, I_h FE Interpolation, then

$$\| u - I_h u \|_{m,p,\Omega} \leq C h^{k+1-m} |u|_{k+1,p,\Omega}$$

($*$ required to define γ on $H^{k+1,p}(\Omega)$)

(without pf here)

1.15 Lemma (Céa)

V Hilbertspace, $V_{h,0} \subset V$ closed subspace, $\ell \in V'$, $a(.,.)$ bd., coercive bilinear form,

$\left(a(u,u) \geq c_0 |u|^2,\ a(u,v) \leq C_1 |u||v| \text{ with } 0 \leq c_0 \leq C_1 < \infty \right)$

then for u, u_h with
$$a(u, \varphi) = \ell(\varphi)\ \forall \varphi \in V$$
$$a(u_h, \varphi_h) = \ell(\varphi_h)\ \forall \varphi_h \in V_h$$

$$\| u - u_h \|_V \leq \frac{C_1}{c_0} \inf_{v_h \in V_{h,0}} \| u - v_h \|_V .$$

Pf

$$\| u - u_h \|_V^2 \leq \frac{1}{c_0} a(u - u_h, u - u_h)$$
$$= \frac{1}{c_0} a(u - u_h, u - v_h) + \underbrace{a(u - u_h, v_h - u_h)}_{\ell(v_h - u_h) - \ell(v_h - u_h) = 0}$$
$$\leq \frac{C_1}{c_0} \| u - u_h \|_V \| u - v_h \|_V$$

$\square$

1.16 **Thm** (convergence of the FEM)

Assume $u \in H^{k+1,2}(\Omega)$ is a weak solution of $(\tilde{P}_L)$ and $u_h = u_h^0 + u_h^\partial \in V_h$ a solution of $(\tilde{P}_{h,L})$, $u_h^\partial = \mathcal{I}_h u^\partial$ on $\partial\Omega$, then
$$\|u - u_h\|_{1,2,\Omega} \leq C h^k |u|_{k+1,2,\Omega}$$

Notation: $|u|_{k+1,2,\Omega} = \left(\sum_{|\alpha|=k+1} \|\partial_\alpha u\|_2^2 \right)^{\frac{1}{2}}$ (semi norm)

Pf $\boxed{u^\partial = u_h^\partial = 0}$ 1.24 ⊕ 1.25

$\boxed{\text{general case}}$ $u^0 = u - u^\partial \qquad u_h^0 = u_h - u_h^\partial$

$$a(u_h^0, \varphi_h) = \int_\Omega f \varphi_h - a(u_h^\partial, \varphi_h) \quad \forall \varphi_h \in V_{h,0}$$

$$w_h := u_h^0 + u_h^\partial - \mathcal{I}_h u = u_h - \mathcal{I}_h u \in V_{h,0}$$

$$\Rightarrow \quad a(w_h, \varphi_h) = \int_\Omega f \varphi_h \, dx - a(\mathcal{I}_h u, \varphi_h) \quad \forall \varphi_h \in V_{h,0}$$

$$\ominus \quad a(0, \varphi_h) = \int_\Omega f \varphi \, dx - a(u, \varphi) \quad \forall \varphi \in H_0^{1,2}(\Omega)$$

$(\varphi = \varphi_h)$

$$a(w_h, \varphi_h) = a(u - \mathcal{I}_h u, \varphi_h) \quad \forall \varphi \in V_{h,0}$$

$$\Rightarrow \quad \|w_h\|_{1,2}^2 \leq \frac{1}{c_0} a(w_h, w_h) = \frac{1}{c_0} a(u - \mathcal{I}_h u, w_h)$$
$$\leq \frac{C_1}{c_0} \|u - \mathcal{I}_h u\|_{1,2} \|w_h\|_{1,2}$$

$$\Rightarrow \quad \|w_h\|_{1,2} \leq \frac{C_1}{c_0} \|u - \mathcal{I}_h u\|_{1,2}$$

$$\|u - u_h\|_{1,2} = \|u - \mathcal{I}_h u - w_h\|_{1,2}$$
$$\leq \|u - \mathcal{I}_h u\|_{1,2} + \frac{C_1}{c_0} \|u - \mathcal{I}_h u\|_{1,2} \leq \left(1 + \frac{C_1}{c_0}\right) \|u - \mathcal{I}_h u\|_{1,2}$$
$$\leq C h^k |u|_{k+1,2} \qquad \square$$

1.17. **Lemma** (Aubin-Nitsche)

Assumptions as in 1.16., $u^\partial = u_h^\partial = 0$, then

$$\|u-u_h\|_{0,2,\Omega} \leq \|u-u_h\|_{1,2,\Omega} \cdot \sup_{0\neq g\in L^2(\Omega)} \inf_{\varphi_h\in V_{h,0}} \frac{\|\varphi_g-\varphi_h\|_{1,2,\Omega}}{\|g\|_{0,2,\Omega}}$$

where φ_g is the solution of the dual pb $a(w,\varphi_g) = (w,g) \quad \forall w\in H_0^{1,2}(\Omega)$

Pf

$$\|u-u_h\|_{0,2} \underset{\boxed{\text{Cauchy-Schwarz}}}{=} \sup_{g\in L^2} \frac{(u-u_h,g)}{\|g\|_{0,2}} = \sup_{g\in L^2} \frac{a(u-u_h,\varphi_g)}{\|g\|_{0,2}}$$

$$\underset{\boxed{\text{orthogonality of the error}}}{=} \sup_{g\in L^2} \inf_{\varphi_h\in V_{h,0}} \frac{a(u-u_h,\varphi_g-\varphi_h)}{\|g\|_{0,2}} \qquad \square$$

1.18 **Thm** (L^2 error estimates) Let $a(u,v) = \int_\Omega a\nabla u\nabla v\,dx$, and for the Assumption as in 1.16, $d\leq 3$, for all $g\in L^2(\Omega)$ we have that $\varphi_g \in H^{2,2}(\Omega)$ with $\|\varphi_g\|_{2,2,\Omega} \leq C\|g\|_{0,2,\Omega}$ (*), $\|u-u_h\|_{0,2,\partial\Omega} \leq Ch^{k+1}$

$a_{ij} \in H^{1,\infty}(\Omega)$ $\forall_{i,j\in\{1,\dots d\}}$, then

$$\|u-u_h\|_{0,2,\Omega} \leq Ch^{k+1}\left(|u|_{k+1,2,\Omega}+1\right)$$

Remark Ω convex $\Rightarrow$ dual pb regular (*)

Pf $\boxed{\text{first in the simple case: } u^\partial = u_h^\partial = 0}$

$d\leq 3 \Rightarrow H^{2,2}(\Omega) \hookrightarrow C^0(\bar\Omega)\ (2-\frac{d}{2}>0) \Rightarrow I_h$ well defined on $H^{2,2}(\Omega)$

$$\|\varphi_g - I_h\varphi_g\|_{1,2,\Omega} \underset{1.14}{\leq} Ch|\varphi_g|_{2,2,\Omega} \leq Ch\|g\|_{0,2,\Omega}$$

$$\Rightarrow \|u-u_h\|_{0,2,\Omega} \leq Ch^k|u|_{k+1,2,\Omega}\, h$$

$\boxed{\text{now: the general case}}$

$$w_h := u_h^\circ + u_h^\partial - \mathcal{I}_h u = u_h - I_h u \in V_{h,0} \quad (\text{see pf. of } 1.16)$$

$$\|w_h\|_{0,2}^2 = a(w_h, \varphi_{w_h}) = a(w_h, \varphi_{w_h} - I_h \varphi_{w_h}) + a(w_h, I_h \varphi_{w_h})$$

$$\underset{\text{as in pf 1.16}}{\leq} C \|w_h\|_{1,2,\Omega}\, h \underbrace{\|\varphi_h\|_{2,2,\Omega}}_{\leq C \|w_h\|_{0,2,\Omega}} + \underbrace{a(u - I_h u, I_h \varphi_{w_h})}$$

$$= a(u_h - I_h u, I_h \varphi_{w_h}) + a(u_h - I_h \cancel{u}, \varphi_{w_h})$$

$$\leq C \underbrace{\left(\|u - u_h\|_{1,2} + \|u - I_h u\|_{1,2}\right)}\, h \|w\|_{0,2} + \int_\Omega a \nabla(u - I_h u) \nabla \varphi_{w_h}\, dx$$

$$\underset{1.14,\,1.16}{\leq} C h^k |u|_{k+1,2,\Omega} - \int_\Omega \mathrm{div}(a^T \nabla \varphi_{w_h})(u - I_h u)\, dx + \int_{\partial\Omega} a^T \nabla \varphi_{w_h} \cdot n_{\partial\Omega} (u - I_h u)\, da$$

$$\leq C h^{k+1} |u|_{k+1,2,\Omega} \|w_h\|_{0,2} + C \|a\|_{1,\infty,\Omega} \underbrace{\|\varphi_{w_h}\|_{1,2,\Omega}}_{\underbrace{\|\varphi_{w_h}\|_{2,2,\Omega}}_{\leq C \|w_h\|_{0,2,\Omega}}} \underbrace{\|u - I_h u\|_{0,2,\Omega}}_{\leq C h^{k+1} |u|_{k+1,2,\Omega}}$$

$$+ C \|a\|_{\infty,\partial\Omega} \underbrace{\|\varphi_{w_h}\|_{1,2,\partial\Omega}}_{\overset{\text{trace thm}}{\leq} C \underbrace{\|\varphi_{w_h}\|_{2,2,\Omega}}_{\leq C \|w_h\|_{0,2,\Omega}}} \underbrace{\|u - I_h u\|_{0,2,\partial\Omega}}_{\leq C h^{k+1}}$$

$$\leq C h^{k+1} \left(|u|_{k+1,2,\Omega} + 1\right) \|w_h\|_{0,2,\Omega}$$

$$\Rightarrow \|w_h\|_{0,2,\Omega} \leq C h^{k+1} \left(|u|_{k+1,2,\Omega} + 1\right)$$

$$\Rightarrow \quad \|u-u_h\|_{0,2,\Omega} \leq \|u-I_h u\|_{0,2,\Omega} + \|\underbrace{I_h u - u_h}_{=-w_h}\|_{0,2,\Omega}$$

$$\underset{1.14.}{\leq} C\, h^{k+1} \left(|u|_{k+1,2,\Omega} + 1 \right)$$

$\square$

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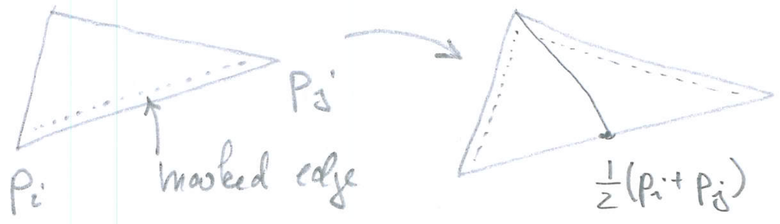
2. A posteriori error estimates and adaptive meshes

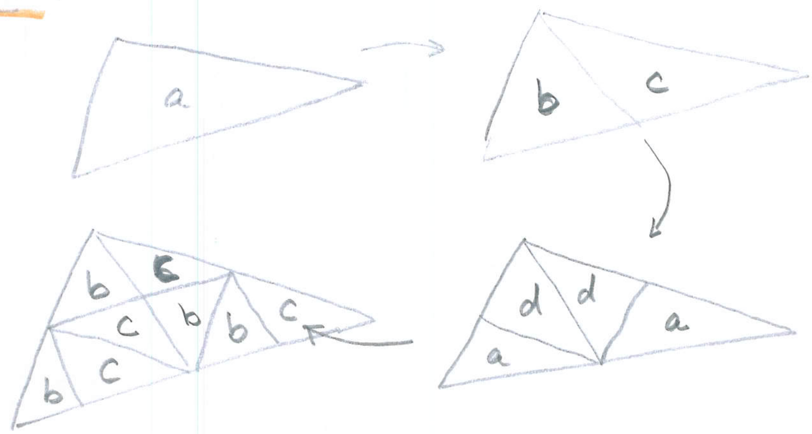
First we study the construction of adaptive meshes:

adaptive simplicial meshes based on bisection in 2D

building block: triangle bisection bisect(T)



recursive application:



$\rightsquigarrow$ triangulations stay regular

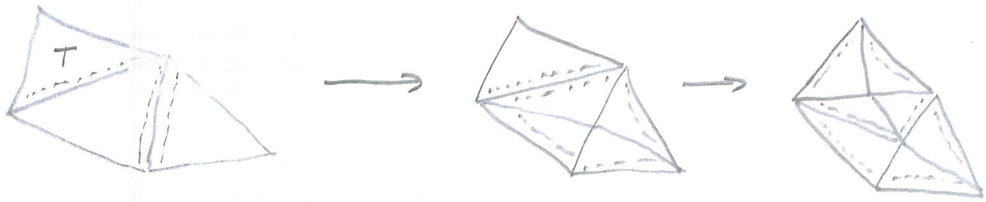
algorithm

notation: nb(T) adjacent simplex at the refinement edge

```
refine(T) {
    if (nb(T) = NULL) bisect(T);
    else { if (T ≠ nb(nb(T)) refine(nb(T));
        bisect(T); bisect(nb(T));
    }
}
```

$\boxed{nb(nb(T)) = T}$

$\boxed{nb(nb(T)) \neq T}$



2.1. Lemma

> if initially the longest edge is marked on every triangle, then the refinement algorithm always terminates.

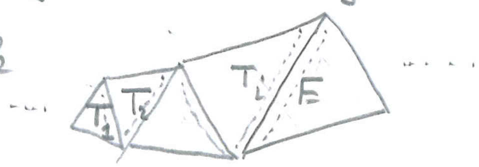
Pf (i) no cycle on the macrogrid: assume that there is a cycle $T_1, — T_m$ with $nb(T_i) = T_{i+1}$ and $nb(T_m) = T_1$

suppose $E \in \mathcal{E}(T_j)$ to be the longest edge of all triangles

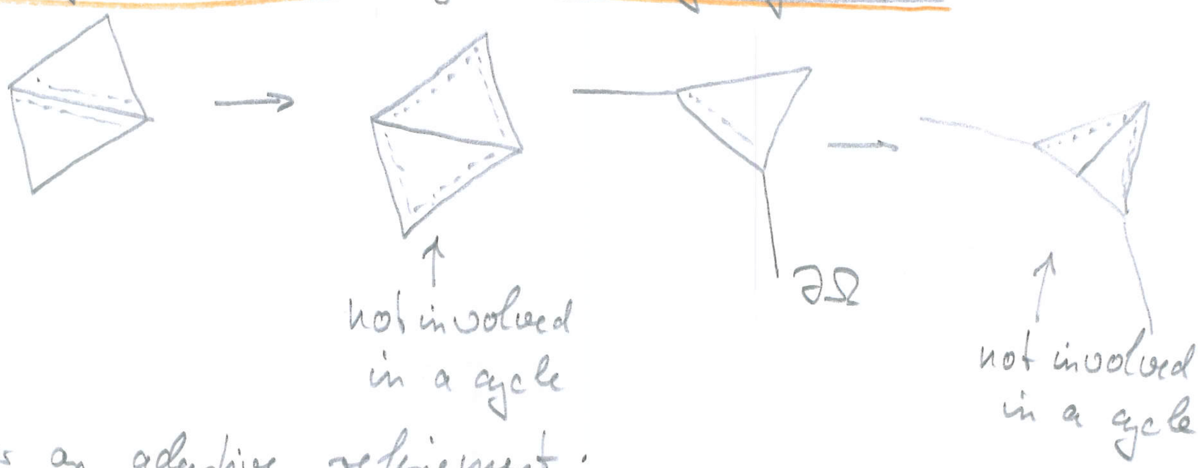
edge set

$T_1, — T_m \Rightarrow E$ is the longest edge in $\mathcal{E}(nb(T_j))$

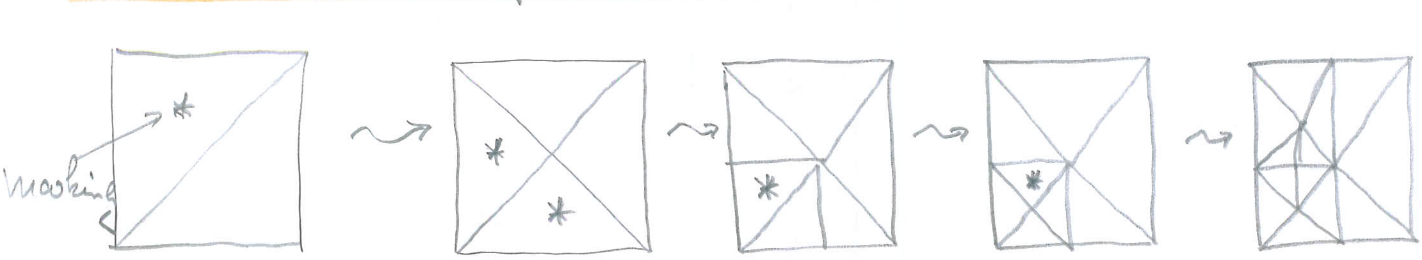
$\Rightarrow nb(nb(T_i)) = T_i$ ↯ no cycle



(ii) every cycle free triangulation stays cycle free:



example for an adaptive refinement:



□

Now: How to define a local and a posteriori error estimator?

Some useful notation:

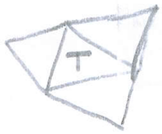
$\mathcal{E}(T)$ faces of T ($d=2$ edges), $\mathcal{E}_h^{(0)} = \bigcup_{T \in \mathcal{T}_h} \mathcal{E}^{(0)}(T)$

$\mathcal{E}^0(T) = \mathcal{E}(T) \setminus \{ E \in \mathcal{E}(T) \mid E \subset \partial\Omega \}$

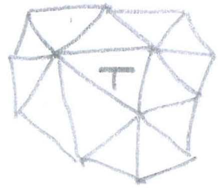
$h_E = \operatorname{diam}(E)$

$h_T = \operatorname{diam}(T)$

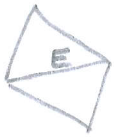
$$\omega_T = \bigcup_{\mathcal{E}(T) \cap \mathcal{E}(T') \neq \emptyset} T'$$



$$\tilde{\omega}_T = \bigcup_{T' \cap T \neq \emptyset} T'$$



$$\omega_E = \bigcup_{E \in \mathcal{E}(T)} T$$



$$\tilde{\omega}_E = \bigcup_{E \cap T \neq \emptyset}$$

$$[\varphi]_E(x) = \lim_{\epsilon \to 0+0} \left(\varphi(x + \epsilon n_E) - \varphi(x - \epsilon n_E) \right)$$

(Jump operator)

requires a unique choice of a normal

ansatz for a residual a posteriori error estimator:

V Banach space, $L \in L(V, V')$ coercive, bd:

$$c \|v\|_V \le \|Lv\|_{V'} \le C_1 \|v\|_V$$

u solves $Lu = \ell$ for $\ell \in V'$

$$\Rightarrow \quad \frac{1}{C_1} \|\overbrace{L\tilde{u} - \underset{\substack{\| \\ Lu}}{\ell}}^{\text{residual}}\|_{V'} \le \|\tilde{u} - u\|_V \le \frac{1}{c} \|\overbrace{L\tilde{u} - \ell}^{\text{residual}}\|_{V'}$$

$\tilde{u}$: approximate solution

"$\frac{1}{C_1}$ residual $\le$ error $\le \frac{1}{c}$ residual"

application: $V = H_0^{1,2}(\Omega)$, $V' = H^{-1,2}(\Omega)$

$$\langle Lu, v \rangle_{V',V} = \int_\Omega a \nabla u \cdot \nabla v \, dx,$$

$$\langle \ell, v \rangle_{V',V} = \int_\Omega f v \, dx - \langle L u^\partial, v \rangle_{V',V}$$

$u \to u^0 \qquad \tilde{u} \to u_h^0$

$$\boxed{u^\partial = u_h^\partial} \quad \frac{1}{C} \sup_{\varphi \in V} \Big(\underbrace{a(u_h^0, \varphi) - \ell(\varphi)}_{a(u_h, \varphi) - \int_\Omega f \varphi \, dx} \Big)$$

$$\leq \|u - u_h\|_{1,2} \leq \frac{1}{c} \sup_{\varphi \in V} \Big(\underline{a(u_h, \varphi) - \int_\Omega f \varphi \, dx} \Big)$$

q: How to estimate this term

how we make this estimate concrete for Lagrangian FEs :

2.2. Lemma (local interpolation estimate)

There exists $\mathcal{I}_h : H^{1,2}(\Omega) \to \mathcal{V}_h$ on a regular triangulation with

$$\|\varphi - \mathcal{I}_h \varphi\|_{0,2,T} \leq C_1 \, h(T) \, |\varphi|_{1,2,\tilde{\omega}_T}$$

$$\|\varphi - \mathcal{I}_h \varphi\|_{0,2,E} \leq C_2 \, h(E)^{\frac{1}{2}} \, |\varphi|_{1,2,\tilde{\omega}_E}$$

Furthermore, there exists $\mathcal{I}_h^0 : H_0^{1,2}(\Omega) \to \mathcal{V}_{h,0}$ with the same estimates $\forall \varphi \in H_0^{1,2}(\Omega)$.

Here $\mathcal{V}_h$ is a Lagrangian FE space.

Pf (later)

this this tool for $f \neq f_h$, $u^\partial \neq u_h^\partial$:

$(\tilde{P})$ $\quad a(u, \varphi) = (f, \varphi) \;\; \forall \varphi \in H_0^{1,2}, \quad u = u^\partial$ on $\partial\Omega$

$(\tilde{P}_h)$ $\quad a(u_h, \varphi_h) = (f_h, \varphi_h) \;\; \forall \varphi_h \in \mathcal{V}_{h,0}, \quad u_h = u_h^\partial$ on $\partial\Omega$

with $$a(u, v) = \int_\Omega \underbrace{\sum_{i,j} a_{ij} \, \partial_j u \, \partial_i v}_{a \nabla u \cdot \nabla v} \, dx$$

$$a(u-u_h,\varphi) \underset{\underset{H_0^{1,2}(\Omega)}{\uparrow}}{} = (f,\varphi) - a(u_h,\varphi)$$

$$\underset{(\tilde{P}_h)}{=} (f-f_h,\varphi) + (f_h,\varphi-\varphi_h) - a(u_h,\varphi-\varphi_h) \qquad \underset{\varphi_h}{\overset{H_0^{1,2}(\Omega)}{\in}}$$

$$= \sum_{T\in \mathcal{J}_h} \int_T (f-f_h)\varphi + f_h(\varphi-\varphi_h) - a\nabla u_h \nabla(\varphi-\varphi_h)\,dx$$

integration by parts

$$= \sum_{T\in \mathcal{J}_h} \int_T (f-f_h)\varphi + \big(f_h + \operatorname{div}(a\nabla u_h)\big)(\varphi-\varphi_h)\,dx + \sum_{E\in\mathcal{E}^\circ(T)} \int_E a\nabla u_h\cdot \underbrace{n_T}_{\text{outer normal of } T}(\varphi-\varphi_h)\,da$$

$$= \sum_{T\in \mathcal{J}_h} \int_\Omega (f-f_h)\varphi + \big(f_h + \operatorname{div}(a\nabla u_h)\big)(\varphi-\varphi_h)\,dx + \frac{1}{2}\sum_{E\in\mathcal{E}^\circ(T)} \int_E [a\nabla u_h\cdot n_T]_E \underbrace{(-n_E\cdot n_T)}_{=\begin{cases}+1; & n_E\neq n_T\\ -1; & n_E = n_T\end{cases}}(\varphi-\varphi_h)\,da$$

spreading edge contribution equally on ω_E

$$\leq \sum_{T\in \mathcal{J}_h}\Big(\|f-f_h\|_{0,2,T}\|\varphi\|_{0,2,T} + \|f_h + \operatorname{div}(a\nabla u_h)\|_{0,2,T}\|\varphi-\varphi_h\|_{0,2,T} + \frac{1}{2}\sum_{E\in\mathcal{E}^\circ(T)} \|[a\nabla u_h\cdot n_E]_E\|_{0,2,E}\,\|\varphi-\varphi_h\|_{0,2,E}\Big)$$

$\varphi_h = \mathring{\mathcal{I}}_h\varphi$

$$\underset{(2.2)}{\leq} \sum_{T\in \mathcal{J}_h}\Big(\|f-f_h\|_{0,2,T}\|\varphi\|_{0,2,T} + C_1\overbrace{\|h_T(\operatorname{div}(a\nabla u_h)+f_h)\|_{0,2,T}}^{\text{element residual}}\|\varphi\|_{1,2,\tilde{\omega}_T} + \frac{C_2}{2}\sum_{E\in\mathcal{E}^\circ(T)} \underbrace{\|h_E^{\frac{1}{2}}[a\nabla u_h\cdot n_E]_E\|_{0,2,E}}_{\text{singular residual}}\underbrace{\|\varphi\|_{1,2,\tilde{\omega}_E}}_{\leq \|\varphi\|_{1,2,\hat{\omega}_T}}\Big)$$

$$\underset{\substack{\text{Cauchy Schwarz}\\ \text{in } \mathbb{R}^{\#\mathcal{J}_h}}}{\leq} C_P\|f-f_h\|_{0,2,\Omega}\|\varphi\|_{1,2,\Omega} + \max\Big(C_1,\frac{C_2}{2}\Big)\Big[\sum_{T\in\mathcal{J}_h}\Big(\|h_T(\operatorname{div}(a\nabla u_h)+f_h\|^2_{0,2,T} + \sum_{E\in\mathcal{E}^\circ(T)}\|h_E^{\frac{1}{2}}[a\nabla u_h\cdot n_E]_E\|^2_{0,2,E}\Big]^{\frac{1}{2}}$$

$$\cdot\Big[\sum_{T\in\mathcal{J}_h}\|\varphi\|^2_{1,2,\tilde{\omega}_T}\Big]^{\frac{1}{2}}\Big\} \leq \tilde{C}\,\|\varphi\|_{1,2,\Omega}$$

where $\tilde{C}$ depends on the regularity of $\mathcal{J}_h$:

$$\tilde{C} = \sqrt{\max_{T \in \mathcal{J}_h} \# \{ T' \in \mathcal{J}_h \mid T \subset \tilde{\omega}_{T'} \}}$$

with $\eta_T := \left[\| h_T (\operatorname{div}(a \nabla u_h) + f_h \|^2_{0,2,T} + \sum_{E \in \mathcal{E}^\circ(T)} \| h_E^{\frac{1}{2}} [a \nabla u_h \cdot n_E] \|^2_{0,2,E} \right]^{\frac{1}{2}}$

we obtain

$$a(u - u_h, \varphi) \leq \left(\|f - f_h\|_{0,2,\Omega} + \underbrace{\tilde{C} \max\left(C_1, \tfrac{C_2}{2}\right)}_{=: \hat{C}} \left(\sum_{T \in \mathcal{J}_h} \eta_T^2 \right)^{\frac{1}{2}} \right) \|\varphi\|_{1,2,\Omega}$$

for $u^\partial = u_h^\partial$:

$$\underbrace{|u - u_h|}_{\in H_0^{1,2}}{}_{1,2,\Omega} \leq \frac{1}{c} \sup_{\|\varphi\|_{1,2,\Omega} = 1,\ \varphi \in H_0^{1,2}(\Omega)} a(u - u_h, \varphi)$$

$$\leq \frac{1}{c} \left(C_p \|f - f_h\| + \hat{C} \left(\sum_{T \in \mathcal{J}_h} \eta_T^2 \right)^{\frac{1}{2}} \right)$$

for general bd. data:

$$|(u - u_h) - (u^\partial - u_h^\partial)|^2_{1,2,\Omega} \leq \frac{1}{c} a(u - u_h - (u^\partial - u_h^\partial), \underbrace{u - u_h - (u^\partial - u_h^\partial)}_{\in H_0^{1,2}(\Omega)})$$

$$\leq \frac{1}{c} \left(C_p \|f - f_h\|_{0,2,\Omega} + \hat{C} \left(\sum_{T \in \mathcal{J}_h} \eta_T^2 \right)^{\frac{1}{2}} + C |u^\partial - u_h^\partial|_{1,2,\Omega} \right) \cdot |(u - u_h) - (u^\partial - u_h^\partial)|_{1,2,\Omega}$$

$\Rightarrow$

$$\boxed{\begin{array}{l} \textbf{2.3. Thm} \text{ (residual a posteriori error estimate, robustness)} \\ u, u_h \text{ solution of } (\tilde{P}), (\tilde{P}_h) \text{ with } f, f_h \in L^2(\Omega), u^\partial, u_h^\partial \in H^{1,2}(\Omega), \\ \text{then } \quad |u - u_h|_{1,2,\Omega} \leq \dfrac{C_p}{c} \|f - f_h\|_{0,2,\Omega} + \dfrac{\hat{C}}{c} \left(\sum_{T \in \mathcal{J}_h} \eta_T^2 \right)^{\frac{1}{2}} + \left(1 + \dfrac{C}{c}\right) |u^\partial - u_h^\partial|_{1,2,\Omega} \\ \text{where } \quad \eta_T := \left[\| h_T (\operatorname{div}(a \nabla u_h) + f_h) \|^2_{0,2,T} + \sum_{E \in \mathcal{E}^\circ(T)} \| h_E^{\frac{1}{2}} [a \nabla u_h \cdot n_E]_E \|^2_{0,2,E} \right]^{\frac{1}{2}} \end{array}}$$

robustness $\longleftrightarrow$ error estimated by the error estimator

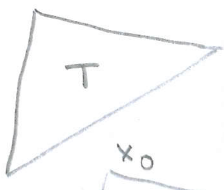
q: can we also estimate the local error estimator by the local error $\longleftrightarrow$ efficiency (if we refine we do it at the "right" position: large error estimator $\Rightarrow$ large error)

2.4. **Thm** (efficiency)

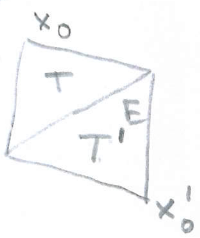
Assumptions as in 2.3., f_ℓ, a_{ij} p.w. polynomials, then

$$\eta_T \le C \left(|u-u_\ell|^2_{1,2,\omega_T} + h(T)^2 \|f-f_\ell\|^2_{0,2,\omega_T} \right)^{\frac{1}{2}}$$

At first: construction of bubble fct's:



$$b_T := \prod_{j=0}^{d} \lambda_j \in \mathcal{P}_{d+1} \quad (\operatorname{supp} b_T = T)$$



$E = T \cap T'$ face ((d-1) dim. sub simplex)

without restriction let $x_0 \in T$, $x_0' \in T'$ be the vertices opposite to E in both simplices

$$b_E(x) := \begin{cases} \lambda_1(x) \cdots \lambda_d(x); & x \in T \\ \lambda_1'(x) \cdots \lambda_d'(x); & x \in T' \end{cases}$$

$$b_E|_T,\ b_E|_{T'} \in \mathcal{P}_d$$

$$\boxed{\varphi_{b,T} := \varphi\, b_T \quad , \quad \psi_{b,E} := \psi\, b_E}$$

Now, we proof some scaling estimates and estimates for the bubble functions!

To this end, we first study the scaling of Sobolev norms:

2.5 Lemma $\hat T = \mathrm{conv}(0, e_1, \dots, e_d)$ reference simplex,

$T = F(\hat T)$, $F(x) = Ax + x_0$, $A = \left(x_1 - x_0 \;\cdots\; x_d - x_0 \right)$,

$\hat u = u \circ F$, $u \in H^{m,p}(T)$, then

$$|\hat u|_{m,p,\hat T} \le C\, h(T)^m \varrho(T)^{-\frac{d}{p}} |u|_{m,p,T} \qquad (m \ge 0)$$

$$|u|_{m,p,T} \le C\, \varrho(T)^{-m} h(T)^{\frac{d}{p}} |\hat u|_{m,p,\hat T}$$

$$|u|_{0,p,E \in \mathcal{E}(T)} \le C\, h(E)^{\frac{d-1}{p}} |\hat u|_{0,p,\hat E = F^{-1}(E)}$$

$$|u|_{0,p,\hat E} \le C\, \varrho(E)^{-\frac{d-1}{p}} |u|_{0,p,E}$$

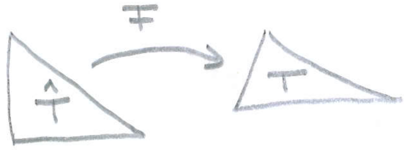
Pf $\partial_{\hat x_i} \hat u = \partial_{\hat x_i}(u \circ F)(\hat x) = \sum_{j=1}^{d} (\partial_{x_j} u) \circ F\, A_{ji} \Rightarrow$

$$|\partial^\alpha_{\hat x} \hat u(\hat x)| \le C\, |||A|||^{|\alpha|} \max_{|\beta| = |\alpha|} |\partial^\alpha_x u(x)|$$

$$\Rightarrow |\hat u|_{m,p,\hat T} \le C\, |||A|||^m \max_{|\beta|=|\alpha|} \underbrace{\left(\int_{\hat T} |\partial^\beta_x u \circ F|^p \, d\hat x \right)^{\frac{1}{p}}}_{\left(\int_T |\partial^\beta_x u|^p \det A^{-1} \, dx \right)^{\frac{1}{p}}}$$

$$\le C\, |||A|||^m (\det A)^{-\frac{1}{p}} |u|_{m,p,T}$$

now:



For $\hat\tau \in \mathbb{R}^d$ with $|\hat\tau| = 1$ $\exists\, \hat x, \hat y \in \hat T$ with $\hat y = \hat x + \varrho(\hat T)\hat\tau$ $\quad (\varrho(\hat T) = |\hat y - \hat x|)$

$$\Rightarrow \frac{|A(\hat x - \hat y)|}{|\hat x - \hat y|} = \frac{|A\hat x - A\hat y|}{|\varrho(\hat T)|} \le \frac{h(T)}{\varrho(\hat T)}$$

$$\Rightarrow \boxed{|||A||| \le \frac{h(T)}{\varrho(\hat T)}} \quad \text{in analogy} \quad \boxed{|||A^{-1}||| \le \frac{h(\hat T)}{\varrho(T)}}$$

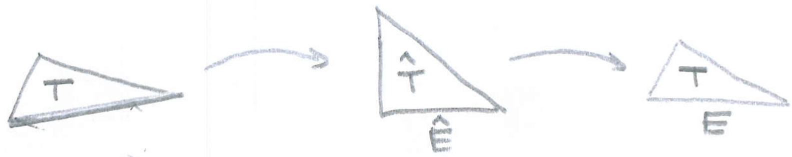
furthermore
$$\boxed{|\det A| \leq |||A|||^d = \frac{h(T)^d}{\varrho(\hat T)^d}, \quad |\det A^{-1}| \leq \frac{h(\hat T)^d}{\varrho(T)^d} \qquad |\det A| = \frac{1}{|\det A^{-1}|} \geq \frac{\varrho(T)^d}{h(\hat T)^d}}$$

Thus, we get
$$|\hat u|_{m,p,\hat T} \leq \frac{h(T)^m}{\varrho(\hat T)^m} \frac{h(\hat T)^{\frac{d}{p}}}{\varrho(T)^{\frac{d}{p}}} |u|_{m,p,T}$$
$$\leq \underbrace{\varrho(\hat T)^{-m}}_{=C} \underbrace{h(\hat T)^{\frac{d}{p}}}_{\leq C} h(T)^m \varrho(T)^{-\frac{d}{p}} |u|_{m,p,T}$$

reverse estimate analogously!

estimates for $|\hat u|_{0,p,\hat E}$, $|u|_{0,p,E}$ based on subsimplex transform $\square$

now we also can estimate:



$$|p_{b,E}|^2_{m,2,\omega_E} = \sum_{T\subset\omega_E} |p_{b,E}|^2_{m,2,T}$$
$$\underset{\text{scaling}}{\leq} \sum_{T\subset\omega_E} \underbrace{|\hat p_{b,\hat E}|^2_{m,2,\hat T}}_{\leq \|\hat p_{b,\hat E}\|^2_{m,2,\hat T}} \underbrace{h(T)^d}_{\leq C h(E)} \underbrace{\varrho(T)^{-2m}}_{\leq C h(E)^{-2m}}$$
$$\underset{\boxed{\text{norm equiv. in finite dimension}}}{\leq} C \sum_{T\subset\omega_E} \|\hat p_{b,\hat E}\|^2_{0,2,E}\, h(E)^{d-2m}$$

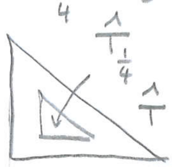
$p \in \mathcal{P}_M$
$$\underset{\boxed{\text{norm equiv see below}}}{\leq} C(M) \|\hat p\|^2_{0,2,\hat E}\, h(E)^{d-2m}$$
$$\underset{\text{Scaling}}{\leq} C(M) \|p\|^2_{0,2,E} \underbrace{\varrho(E)^{d-1}}_{\leq h(E)^{d-1}} h(E)^{d-2m}$$
$$\leq C(M) h(E)^{1-2m} \|p\|^2_{0,2,E}$$

to summarize:
$$\boxed{|p_{b,E}|_{m,2,\omega_E} \leq C h(E)^{\frac{1}{2}-m} |p|_{0,2,E}}$$

furthermore for $p \in \mathbb{P}_M$:

$$\|p\|^2_{0,2,T} \overset{\text{scaling}}{\le} C h^d \|\hat p\|^2_{0,2,\hat T} \underset{\text{norm equiv}}{\le} C h^d \Big(\max_{|\alpha|\le M} |\partial^\alpha \hat p(\underbrace{s_{\hat T}}_{\text{centre of mass}})|\Big)^2$$

$$\underset{\text{norm equiv.}}{\le} C h^d \|\hat p\|^2_{0,2,\hat T_{\frac14}} \qquad \Big(\text{with } \hat T_{\frac14} = \{\hat x \in \hat T \mid \operatorname{dist}(\hat x, \partial \hat T) > \tfrac14\}\Big)$$



$$\le C h^d \int_{\hat T} \hat p\, \hat p_{b,\hat T}\, dx$$

$$\overset{\text{scaling}}{\le} C \int_T p\, p_{b,T}\, dx$$

(in analogy for $\|p\|^2_{0,2,E}$)

$$\boxed{\begin{aligned} \|p\|^2_{0,2,T} &\le C(M) \int_T p\, p_{b,T}\, dx \\ \|p\|^2_{0,2,E} &\le C(M) \int_E p\, p_{b,E}\, dx \end{aligned}}$$

Pf (i) rewriting of the element residual and the singular residual:

(of Thm 2.4) supp $\varphi \subset T$ $\quad \varphi \in H^{1,2}_0(\Omega)$

① $$\int_T \underbrace{(f_h + \operatorname{div}(a \nabla u_h))}_{=:\, p} \varphi\, dx = -a(u_h, \varphi) + (f, \varphi) - (f - f_h, \varphi) = a(u - u_h, \varphi) + (f_h - f, \varphi)$$

supp $\psi \subset \omega_E$, $\psi \in H^{1,2}_0(\Omega)$ $\quad \psi(x) = \psi^\lambda(\lambda_1^{(1)}(x), \dots, \lambda_d^{(1)}(x))$ on $T^{(1)}$:

② $$\int_E \underbrace{[a \nabla u_h \cdot n_E]_E}_{=:\, \tilde p} \psi\, da = \sum_{T \subset \omega_E} \int_T a \nabla u_h \cdot \nabla \psi + \operatorname{div}(a \nabla u_h) \psi\, dx$$

$$= a(u_h - u, \psi) + (f - f_h, \psi) + \sum_{T \subset \omega_E} \int_T (\operatorname{div}(a \nabla u_h) + f_h) \psi\, dx$$

(ii) estimates:

$$\|\underbrace{\operatorname{div}(a \nabla u_h) + f_h}_{=\, p}\|^2_{0,2,T} \le C \int_T (\operatorname{div}(a \nabla u_h) + f_h)\, p_{b,T}$$

$$\overset{①}{\le} C\, |u - u_h|_{1,2,T} \underbrace{|p_{b,T}|_{1,2,T}}_{\le\, C h_{(T)}^{-1} \|p\|_{0,2,T}} + \|f - f_h\|_{0,2,T} \underbrace{\|p_{b,T}\|_{0,2,T}}_{\le\, C \|p\|_{0,2,T}}$$

normequivalenz and scaling

$$\Rightarrow \| h(T)(\operatorname{div}(a\nabla u_h)+f_h)\|_{0,2,T} \le C\left(|u-u_h|_{1,2,T} + h(T)\|f-f_h\|_{0,2,T}\right)$$

$$\|\underbrace{[a\nabla u_h\cdot n_E]_E}_{=\tilde p}\|^2_{0,2,E} \le C\int_E [a\nabla u_h\cdot n_E]_E\,\tilde p_{b,E}\,da$$

$$\overset{②}{=} C\Big(a(u_h-u,\tilde p_{b,E}) + (f-f_h,\tilde p_{b,E}) + \sum_{T\subset\omega_E}\int_T (\operatorname{div}(a\nabla u_h)+f_h)\tilde p_{b,E}\,dx$$

$$\le C\Big(|u-u_h|_{1,2,\omega_E}\underbrace{|\tilde p_{b,E}|_{1,2,\omega_E}}_{\le C h(E)^{-\frac12}\|\tilde p\|_{0,2,\omega_E}} + \|f-f_h\|_{0,2,\omega_E}\underbrace{\|\tilde p_{b,E}\|_{0,2,\omega_E}}_{\le C h^{\frac12}\|\tilde p\|_{0,2,E}}$$

$$+\underbrace{\sum_{T\subset\omega_E}\|\operatorname{div}(a\nabla u_h)+f_h\|_{1,2,T}}_{\underset{\text{see above}}{\le} C\left(|u-u_h|_{1,2,T}+h(T)\|f-f_h\|_{0,2,T}\right)}\cdot\underbrace{\|\tilde p_{b,E}\|_{0,2,\omega_E}}_{\le C h(E)^{-\frac12}\|\tilde p\|_{0,2,E}}$$

$\| p$

$$\Rightarrow \|h^{\frac12}[a\nabla u_h\cdot n_E]\|_{0,2,E} \le C\Big(|u-u_h|_{1,2,\omega_E} + \sum_{T\subset\omega_E} h(T)\|f-f_h\|_{0,2,T}\Big)$$

$$\Rightarrow \quad \eta_T \le C\left(|u-u_h|^2_{1,2,\omega_T} + h(T)^2\|f-f_h\|^2_{0,2,\omega_T}\right)^{\frac12}$$

norm equiv. in $\mathbb{R}^2$
regularity of $\mathcal{T}_h$

□

we still have to prove Lemma 2.2. (local interpolation estimate):

general problem: $H^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\bar{\Omega})$ falls $m - \frac{d}{p} > k+\alpha$ ($\geq$ if $\alpha > 0$)

in particular: $H^{2,2}(\Omega) \hookrightarrow C^{0}(\bar{\Omega})$ for $d \leq 3$

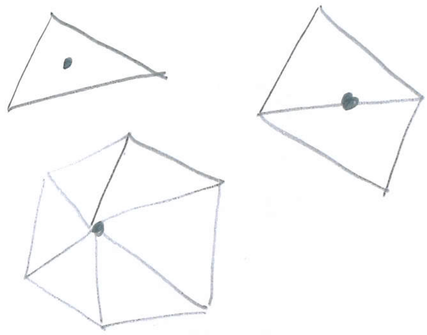
but $H^{1,2}(\Omega) \not\hookrightarrow C^{0}(\bar{\Omega})$ for $d > 1$

hence: Lagrangian interpolation $\mathcal{I}_h u = \sum_{i \in I_h} u(x^i)\varphi_h^i$ not well defined!

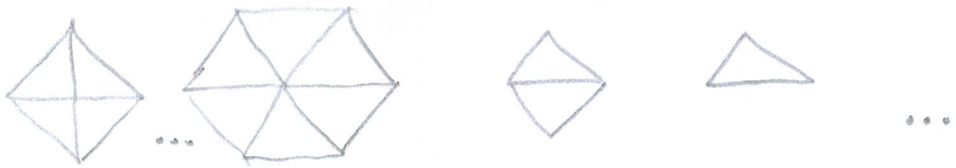
solution: $\boxed{\text{replace functionevaluation by evaluation of a local projection!}}$

notation: $\varphi_h^j \in V_h$ basis fct

$$\omega_j = \operatorname{supp} \varphi_h^j$$



for each topological type we consider a reference support $\hat{\omega}_j$:



Furthermore let $F_j : \hat{\omega}_j \to \omega_j$ be a piecewise affine, bijective, and continuous reference map.

2.6. Definition (local projection operator)

For $\hat{u} \in L^1(\hat{\omega}_j)$ define $\hat{P}_j = \hat{P}_j[\hat{\omega}_j]$ with $\hat{P}_j \hat{u} \in V[\hat{\omega}_j]$, with $V[\hat{\omega}_j]$ the Lagrangian finite element space on $\hat{\omega}_j$, via

$$\int_{\hat{\omega}_j} (\hat{P}_j \hat{u} - \hat{u})\hat{\varphi}\, d\hat{x} = 0 \quad \forall \hat{\varphi} \in V[\hat{\omega}_j]$$

(orthogonal projection of $\hat{u}$ on $V[\hat{\omega}]$ in $L^2(\hat{\omega}_j)$)

For $u \in L^1(\omega_j)$ define $P_j u = \hat{P}_j(u \circ F_j) \circ F_j^{-1} \in V[\omega_j]$

remarks : • $\hat{P}_j$ orthogonal projection, P_j only if $F_i x = Qy + b$ with Q orthogonal

• $u \in H^{1,p}(\omega_j) \Rightarrow P_j u \in H^{1,p}(\omega_j)$

attention: in general not true for $H^{m,p}$ with $m > 1$!

2.7 Definition (global projection)

$\mathcal{I}_k^{(0)} : H^{l,p}(\Omega) (\cap H_0^{1,p}(\Omega)) \to \mathcal{V}_{k(,0)}$ (Lagrangian FE space $\mathcal{V}_{k(,0)}$)

$$\mathcal{I}_k^{(0)}(u)(x) = \sum_{i \in I_k^{(0)}} (P_i u)(\underline{x}_i) \varphi_k^i(x)$$

where $\underline{x}_i$ are the Lagrangian nodes and φ_k^i corresponding Lagrangian basis fct's.

2.8 Proposition (estimate for P_i) If $k, l \in \mathbb{N}_0$, Lagrangian-FE of order k, $k \geq 1$, $0 \leq l \leq k+1$, $1 \leq p \leq \infty$, $l > 1 \Rightarrow l - \frac{d}{p} > 0$, then for $u \in H^{l,p}(\omega_i)$ the estimate

$$|u - P_i u|_{m,p,T} \leq C\, h(T)^{l-m} |u|_{l,p,\omega_i}$$

holds for $T \subset \omega_i$ simplex, $\underline{x}_i \in T$, $m = 0, 1$, $l \geq m$.

Remark • For $p = 2$ one could directly project on $\mathcal{P}_k$:

$$0 = \int_{\omega_i} (\underbrace{P_i u}_{\in \mathcal{P}_k} - u)\, q \, dx \quad \forall q \in \mathcal{P}_k .$$

this does not work for $p \neq 2$!

2.9. Thm (projection estimate)

Assumptions as in 2.8., $u \in H^{l,p}(\Omega)$ ($, l \geq 1, u \in H_0^{1,p}(\Omega)$)

then $$|u - \mathcal{I}_k^{(0)} u|_{m,p,T} \leq C\, h(T)^{l-m} |u|_{l,p,\tilde{\omega}_T}$$

and $$|u - \mathcal{I}_k^{(0)} u|_{m,p,\Omega} \leq C\, h(T)^{l-m} |u|_{l,p,\Omega}$$

Pf (2.8):

$\boxed{m=\ell=0}$ $\quad (\hat u - \hat P_{2}\hat u, \hat\varphi)_{0,2,\hat\omega_2} = 0 \quad \forall \hat\varphi \in \mathcal{V}[\hat\omega_2] \Rightarrow$

$$\|\hat P_2 \hat u\|^2_{0,p,\hat\omega_2} \underset{\uparrow}{\le} C_1 \|\hat P_2 \hat u\|^2_{0,2,\hat\omega_2} \underset{\uparrow}{\le} C_1 \|\hat u\|_{0,p,\hat\omega_2} \|\hat P_2 \hat u\|_{0,p',\hat\omega_2}$$

norm equiv in finite dim

does not work for the standard L^2-proj. (see remark above

$\boxed{(\hat u - \hat P_2 \hat u, \hat P_2 \hat u) = 0 \ \& \ \text{Hölder } \frac{1}{p} + \frac{1}{p'} = 1}$

$$\le C_1 \|\hat u\|_{0,p,\hat\omega_2} \|\hat P_2 \hat u_2\|_{0,p,\hat\omega_2}$$

$$\Rightarrow \|\hat P_2 \hat u\|_{0,p,\hat\omega_2} \le C_1 \|\hat u\|_{0,p,\hat\omega_2}$$

$\underset{\text{scaling argument}}{\Rightarrow}$ $\boxed{\|P_2 u\|_{0,p,\omega_2} \le C_1 \|u\|_{0,p,\omega_2} \quad \text{(stability)}}$

$$\Rightarrow \|u - P_2 u\|_{0,p,\omega_2} \le \|u\|_{0,p,\omega_2} + \|P_2 u\|_{0,p,\omega_2} \le (1+C_1) \|u\|_{0,p,\omega_2}$$

$\boxed{m=0, \ell\ge 1}$

$$\|u - P_2 u\|_{0,p,\omega_2} \overset{\text{scaling}}{\le} C_1 h^{\frac{d}{p}} \|\hat u - \hat P_2 \hat u\|_{0,p,\hat\omega_2}$$

$\boxed{\hat u - \hat P_2 \hat u = (\hat u - \hat v) + \hat P_2(\hat u - \hat v) \ \forall \hat v \in \mathcal{V}[\hat\omega_2]} \leftarrow \boxed{\hat v = \hat P_2 \hat v \ \forall \hat v \in \mathcal{V}[\hat\omega_2]}$

$$\le C_1 (1+C_1) h^{\frac{d}{p}} \inf_{\hat v \in \mathcal{V}[\hat\omega_2]} \|\hat u - \hat v\|_{0,p,\hat\omega_2}$$

$\boxed{\ell=1}$

Poincaré $(\hat v = \fint_{\hat\omega_2} \hat u \, dx)$

$$\Rightarrow \|u - P_2 u\|_{0,p,\omega_2} \le C_1 h^{\frac{d}{p}} |\hat u|_{1,p,\hat\omega_2} \overset{\text{scaling}}{\le} C_1 h |u|_{1,p,\omega_2}$$

$\boxed{\ell>1}$

Lagrange-interpolation (well defined $\Leftarrow H^{\ell,p} \hookrightarrow C^0$)

$$\Rightarrow \|u - P_2 u\|_{0,p,\omega_2} \le C_1 h^{\frac{d}{p}} \|\hat u - \mathcal{I}^L_h \hat u\|_{0,p,\hat\omega_2}$$

$$\overset{1.14}{\le} C_1 h^{\frac{d}{p}} \sum_{\hat T \subset \hat\omega} |\hat u|_{\ell,p,\hat T} \overset{\text{scaling}}{\le} C_1 h^{\ell} |u|_{\ell,p,\omega_2}$$

stability of $\mathcal{I}^L_h$

$\boxed{m=1}$ $\quad |u - P_i u|_{1,p,\omega_i} \overset{\text{scaling}}{\leq} C_1 h^{\frac{d}{p}-1} |\hat{u} - \hat{P}_i \hat{u}|_{1,p,\hat{\omega}_i}$

$$\leq C_1 h^{\frac{d}{p}-1} \inf_{v \in \mathcal{V}[\hat{\omega}]} \Big(|\hat{u} - \hat{v}|_{1,p,\hat{\omega}_i} + \underbrace{|\hat{P}_i \hat{u} - \hat{P}_i \hat{v}|_{1,p,\hat{\omega}_i}}_{\underset{\text{norm equiv}}{\leq} C_1 \|\hat{P}_i \hat{u} - \hat{P}_i \hat{v}\|_{0,p,\hat{\omega}_i} \underset{\text{Stability}}{\leq} C_1 \|\hat{u} - \hat{v}\|_{0,p,\hat{\omega}_i}} \Big)$$

$\boxed{\ell = 1}$

$$\leq C_1 h^{\frac{d}{p}-1} \inf_{v \in \mathcal{V}[\hat{\omega}]} \|\hat{u} - \hat{v}\|_{1,p,\hat{\omega}_i} \overset{\text{Poincaré (see above)}}{\leq} C_1 h^{\frac{d}{p}-1} |\hat{u}|_{1,p,\hat{\omega}_i} \overset{\text{scaling}}{\leq} C_1 |u|_{1,p,\omega_i}$$

$\boxed{\ell > 1}$

$$\leq C_1 h^{\frac{d}{p}-1} \|\hat{u} - \hat{\mathcal{I}}^L \hat{u}\|_{1,p,\hat{\omega}_i} \overset{1.14}{\leq} C_1 h^{\frac{d}{p}-1} \sum_{\hat{T} \in \hat{\omega}} |\hat{u}|_{\ell,p,\hat{T}} \overset{\text{scaling}}{\leq} C_1 h^{\ell-1} |u|_{\ell,p,\omega_i}$$

$\square$

<u>Pf 2.9.:</u> $\quad \boxed{i_0 \in I(T) = \{ i \in I_h \mid \omega_i \cap T \neq \emptyset \}}$

$$u - \mathcal{I}_h u\big|_T = u\big|_T - \sum_{i \in I(T)} (P_{i_0} u)(x_i) \varphi_h^i\big|_T - \sum_{i \in I(T)} \big((P_i u)(x_i) - (P_{i_0} u)(x_i)\big) \varphi_h^i\big|_T$$

$$= (u - P_{i_0} u)\big|_T - \sum_{i \in I(T) \setminus \{i_0\}} \big((P_i u)(x_i) - (P_{i_0} u)(x_i)\big) \varphi_h^i\big|_T$$

$$\Rightarrow \|u - \mathcal{I}_h u\|_{m,p,T} \leq \underbrace{\|u - P_{i_0} u\|_{m,p,T}}_{\leq C_1 h(T)^{\ell-m} |u|_{\ell,p,\tilde{\omega}_T}} + C_1 \|P_i u - P_{i_0} u\|_{0,\infty,T} |\varphi_h^i|_{m,p,T}$$

$$\leq C_1 h(T)^{\ell-m} |u|_{\ell,p,\tilde{\omega}_T} + C_1 \underbrace{\|\hat{P}_i \hat{u} - \hat{P}_{i_0} \hat{u}\|_{0,\infty,\hat{T}}}_{} \, h(T)^{\frac{d}{p}-m} \underbrace{|\hat{\varphi}_h^i|_{m,p,\hat{T}}}_{\leq C_1}$$

$$\underset{\boxed{\text{norm equiv}}}{\leq} C_1 \|\hat{P}_i \hat{u} - \hat{P}_{i_0} \hat{u}\|_{0,p,\hat{T}}$$

$$\leq C_1 \big(\|\hat{u} - \hat{P}_i \hat{u}\|_{0,p,\hat{T}} + \|\hat{u} - \hat{P}_{i_0} \hat{u}\|_{0,p,\hat{T}} \big)$$

$$\leq C_1 h^{-\frac{d}{p}} \underbrace{\|u - P_i u\|_{0,p,T} + \|u - P_{i_0} u\|_{0,p,T}}_{\leq C_1 h(T)^{\ell} (|u|_{\ell,p,\omega_i} + |u|_{\ell,p,\omega_{i_0}})}$$

$$\leq C_1 h(T)^{\ell-m} |u|_{\ell,p,\tilde{\omega}_T}$$

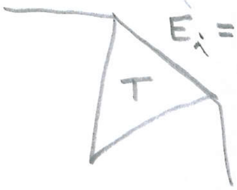
$$\left(\sum_{T\in\mathcal{J}_h}(\cdot)^p\right)^{\frac{1}{p}} \Rightarrow \quad \boxed{\|u-\mathcal{I}_h u\|_{m,p,\Omega} \le C h^{\ell-m}|u|_{\ell,p,\Omega}}$$

now we consider $u\in H_0^{1,p}(\Omega)$ and investigate $\mathcal{I}_h^0$:

$$I(T) = \underset{\text{interior}}{I^0(T)} \cup \underset{\text{bd indizes}}{I^\partial(T)}$$

$$\|u-\mathcal{I}_h^0 u\|_{m,p,T} \le |u-\mathcal{I}_h u|_{m,p,T} + \sum_{i\in I^\partial(T)} |(P_i u)(x_i)|\,\underbrace{|\varphi_h^i|_{m,p,T}}_{\le C h_{(T)}^{\frac{d}{p}-m}}$$

$$\le C h_{(T)}^{\ell-m}|u|_{\ell,p,\tilde\omega_T} + C\sum_{i\in I^\partial(T)} \underbrace{\|\hat P_i \hat u\|_{0,\infty,\hat E_i}}\, h_{(T)}^{\frac{d}{p}-m}$$



E_i = edge containing x_i

$$\underset{\text{norm equiv}}{\le} C\|\hat P_i\hat u - \hat u\|_{0,p,\hat E}$$

$$\underset{\text{trace thm}}{\le} C\|\hat P_i \hat u-\hat u\|_{1,p,\hat T}$$

$$\le C h_{(T)}^{-\frac{d}{p}}\left(\|P_i u-u\|_{0,p,T} + h_{(T)}|P_iu-u|_{1,p,T}\right)$$

$$\le C h_{(T)}^{\ell-\frac{d}{p}}|u|_{\ell,p,\tilde\omega_T}$$

$$\Rightarrow \|u-\mathcal I_h^0 u\|_{m,p,T}\le C h^{\ell-m}|u|_{\ell,p,\tilde\omega_T}$$

$$\left(\sum_{T\in\mathcal{J}_h}(\cdot)^p\right)^{\frac{1}{p}} \Rightarrow \quad \boxed{\|u-\mathcal I_h^0 u\|_{m,p,\Omega}\le Ch^{\ell-m}|u|_{\ell,p,\tilde\omega_T}} \qquad \square$$

Finally we prove Lemma 2.2:

- $\|\varphi-\mathcal I_h^{(0)}\varphi\|_{0,2,T} \overset{2.9}{\le} C_1 h_{(T)}|\varphi|_{1,2,\tilde\omega_T}$ (✓)
- $\|\varphi-\mathcal I_h^{(0)}\varphi\|_{0,2,E} \le C h_{(E)}^{\frac{d-1}{2}} \|\widehat{\varphi-\mathcal I_h^{(0)}\varphi}\|_{0,2,\hat E}$

$$\underset{\text{trace thm}}{\le} C h_{(E)}^{\frac{d-1}{2}}\left(\underbrace{\|\widehat{\varphi-\mathcal I_h^{(0)}\varphi}\|_{0,2,\hat T}}_{\le C h^{-\frac{d}{2}}\|\varphi-\mathcal I_h^{(0)}\varphi\|_{0,2,T}} + \underbrace{|\widehat{\varphi-\mathcal I_h^{(0)}\varphi}|_{1,2,\hat T}}_{\le C h^{-\frac{d}{2}+1}|\varphi-\mathcal I_h^{(0)}\varphi|_{1,2,T}}\right)$$

$$\Rightarrow \|\varphi - \mathcal{I}_h^{(0)}\varphi\|_{0,2,E} \le C\, h(E)^{-\frac{1}{2}} \|\varphi - \mathcal{I}_h^{(0)}\varphi\|_{0,2,T} + C h_{(E)}^{\frac{1}{2}} |\varphi - \mathcal{I}_h^{0}\varphi|_{1,2,T}$$

$$\overset{2,9.}{\le} C\, h(E)^{\frac{1}{2}} |\varphi|_{1,2,\tilde{\omega}_T}$$

$\mathcal{I}_h\varphi|_E$ depends only the basis fct's φ_h^i with $\operatorname{supp}\varphi_h^i \subset \hat{\omega}_E \Rightarrow \tilde{\omega}_T$ can be replaced by $\tilde{\omega}_E$

$\square$

A short review of interpolation estimates for the Lagrangian interpolation:

- $|u - \mathcal{I}_h^L u|_{m,p,\Omega} \overset{!}{\le} C h^{\ell-m} |u|_{\ell,p,\Omega}$ if $P = \mathcal{P}_k$, $\ell \le k+1$, $m = 0,1$, $\ell - \frac{d}{p} > 0$

$\rightarrow$ sufficient to show this on a simplex!

- on the reference simplex $\hat{T}$:

$$|\hat{u} - \hat{\mathcal{I}}^L\hat{u}|_{m,p,\hat{T}} \overset{\hat{\mathcal{I}}q = q}{\le} |\hat{u} - q|_{m,p,\hat{T}} + |\hat{\mathcal{I}}^L(\hat{u} - q)|_{m,p,\hat{T}} \quad (q \in \mathcal{P}_\ell)$$

$$\overset{\hat{\mathcal{I}}^L \in L(H^{\ell,p}(\hat{T}), \mathcal{P}_k)}{\le} C(1 + |||\hat{\mathcal{I}}^L|||)\, \|\hat{u} - q\|_{\ell,p,\hat{T}}$$

[choose $q \in \mathcal{P}_{\ell-1}$ such that $\int_{\hat{T}} \partial^\alpha(\hat{u} - q)\, d\hat{x} = 0 \quad \forall |\alpha| \le \ell - 1$ $\Leftrightarrow$ invertible linear system $\left[\int_{\hat{T}} \partial^\alpha q\, d\hat{x} = 0 \Rightarrow q = 0\right]$

Poincaré (iterative appl.)

$$\Rightarrow \|\hat{u} - q\|_{\ell,p,\hat{T}} \le C\, |\hat{u} - q|_{\ell,p,\hat{T}} = C = |\hat{u}|_{\ell,p,\hat{T}}$$]

$$\Rightarrow \boxed{|\hat{u} - \hat{\mathcal{I}}^L\hat{u}|_{m,p,\hat{T}} \le C\, |\hat{u}|_{\ell,p,\hat{T}}}$$

- scaling: $|u - \mathcal{I}_h^L u|_{m,p,T} \le C\, h(T)^{\frac{d}{p} - m} |\hat{u} - \hat{\mathcal{I}}^L\hat{u}|_{m,p,\hat{T}}$

$$\le C\, h(T)^{\frac{d}{p} - m} |\hat{u}|_{\ell,p,\hat{T}} \le C\, h(T)^{\frac{d}{p} - m}\, h(T)^{-\frac{d}{p} + \ell} |u|_{\ell,p,T}$$

q: which marking strategy leads to a convergent algorithm?

simplification: • (P) $-\Delta u = f$ in $\Omega \subset \mathbb{R}^d$ (bounded) $\boxed{\text{focus: } d=2}$

$u = u^\partial$ on $\partial\Omega$ (polygonal)

• affine FE

grid size fct $= h$ $\quad (h|_{\mathring{T}} = h(T))$

$\mathcal{T}_H$ coarse grid, $\mathcal{T}_h$ fine grid obtained by refinement of $\mathcal{T}_H$

u_H, u_h corresponding solutions

$e_H = u - u_H$, $e_h = u - u_h$

aim: generate via local refinement iteratively a mesh $\mathcal{T}_h$ (grid size fct. h) such that $|e_h|_{1,2} \le \epsilon$ for a given $\epsilon > 0$

assumption: (F) The initial grid $\mathcal{T}_{\bar{H}}$ is sufficiently fine, s.t.

$$\max\{\|f - f_h\|_{0,2,\Omega}, \|h f_h\|_{0,2,\Omega}, |u^\partial - u^\partial_h|_{1,2}\} \le \mu\epsilon$$

$\forall\, \mathcal{T}_h$ refinements of $\mathcal{T}_H$ and a constant $\mu > 0$ to be chosen later.

2.10. Lemma $\mathcal{T}_h$ refinement of $\mathcal{T}_H$ $(H \ge h)$ and $|e_H|_{1,2,\Omega} \ge \frac{\epsilon}{C_H}$,

then
$$|e_H|^2_{1,2,\Omega} \ge |e_h|^2_{1,2,\Omega} + \underbrace{\frac{1}{2}|u_h - u_H|^2_{1,2,\Omega} - C_1(C_H\mu + (C_H\mu)^2)|e_H|^2_{1,2}}_{\text{lower bd of the error}^2\text{ reduction}}$$

Pf
$$|e_H|^2_{1,2} = |\nabla e_h + \nabla(u_h - u_H)|^2_{1,2}$$
$$= |e_h|^2_{1,2} + |u_h - u_H|^2_{1,2} + 2\underbrace{(\nabla e_h, \nabla(u_h - u_H))_{0,2}}$$

$$\boxed{\int_\Omega \nabla e_h \cdot \nabla(u_h - u_H - (u_h^\partial - u_H^\partial))\,dx \overset{(\tilde P),(\tilde P_h)}{=} \int_\Omega (f - f_h)(u_h - u_H - (u_h^\partial - u_H^\partial))\,dx}$$

$$\overset{\downarrow}{=} 2(\nabla e_h, \nabla(u_h^\partial - u_H^\partial))_{0,2} + 2(f - f_h, u_h - u_H - (u_h^\partial - u_H^\partial))_0$$

$$\Rightarrow |e_H|_{1,2}^2 \geq |e_e|_{1,2}^2 + |u_e - u_H|_{1,2}^2 - 2\,|u_e^\partial - u_H^\partial|_{1,2}\left(|e_H|_{1,2} + |u_e - u_H|_{1,2}\right)$$
$$- 2\,C_p \|f - f_e\|_{0,2}\left(|u_e - u_H|_{1,2} + |u_e^\partial - u_H^\partial|_{1,2}\right)$$

$$\overset{\geq}{\uparrow}\; |e_H|_{1,2}^2 + \frac{1}{2}|u_e - u_H|_{1,2}^2 - 4\,|u_e^\partial - u_H^\partial|_{1,2}^2 - 4\,C_p^2 \|f - f_e\|_{0,2}^2$$
$$- 2\,|e_H|_{1,2}\,|u_e^\partial - u_H^\partial|_{1,2} - 2\,C_p \|f - f_e\|_{0,2} \|u_e^\partial - u_H^\partial\|_{1,2}$$

$\boxed{2ab \leq \frac{1}{4}a^2 + 4b^2}$

using (F) $\left(|u_e^\partial - u_H^\partial|_{1,2} \leq |u^\partial - u_e^\partial|_{1,2} + |u^\partial - u_H^\partial|_{1,2} \overset{(F)}{\leq} 2\mu\epsilon \right)$:

$$|e_H|_{1,2}^2 \geq |e_e|_{1,2}^2 + \frac{1}{2}|u_e - u_H|_{1,2}^2 - \left(16\mu^2\epsilon^2 + 4C_p^2\mu^2\epsilon^2 + 4C_p\mu^2\epsilon^2\right)$$
$$- 4\mu\epsilon\,|e_H|_{1,2}$$

$$\geq |e_e|_{1,2}^2 + \frac{1}{2}|u_e - u_H|_{1,2}^2 - \underbrace{\left(16\mu^2 C_H^2 + 4C_p^2\mu^2 C_H^2 + 4C_p\mu^2 C_H^2 + 4\mu C_H\right)}_{\leq C_1 (C_H\mu + C_H^2\mu^2)} \cdot |e_H|_{1,2}^2$$

$\boxed{|e_H|_{1,2} \geq \frac{\epsilon}{C_H}}$ $\boxed{\epsilon \leq C_H |e_H|_{1,2}}$

$\square$

remark $\boxed{f = f_e\,,\ u_e^\partial = u_H^\partial} \Rightarrow$ Galerkin orthogonality

$$(\nabla e_e, \nabla\underbrace{(u_e - u_H)}_{\in V_{H,0}}) = 0$$

$$\Rightarrow |e_H|_{1,2}^2 = |e_e|_{1,2}^2 + |u_e - u_H|_{1,2}^2$$

thus 2.10 is a refined version of the Galerkin orthogonality

Now we consider a set $\mathcal{E}_{ref} \subset \mathcal{E}_H^0$ of refined edges of $\mathcal{T}_H$ in $\mathcal{T}_e$ and investigate the relation of the singular residual on $\mathcal{E}_{ref}$ and the error e_H and the error increment $(u_e - u_H)$.

2.11. Lemma For $\mathcal{E}_{ref} \subset \mathcal{E}_H^\circ$ being refined edges in $\mathcal{T}_h$ we have:

$$\sum_{E\in\mathcal{E}_{ref}} h(E)\, \|[\nabla u_H \cdot n_E]_E\|^2_{0,2,E} \le C_2\, |u_h - u_H|^2_{1,2} + C_3\, \|h f_h\|^2_{0,2}$$

$$\sum_{E\in\mathcal{E}_H^\circ} h(E)\, \|[\nabla u_H \cdot n_E]_E\|^2_{0,2,E} \le C_2\, |e_H|^2_{1,2} + C_3\, \|H f\|^2_{0,2}$$

Pf (compare 2.4.)

we consider discrete bubble fct's:

let $\varphi_E \in V_{h,0}$ basis fct. for the refinement vertex x_E on E

(i) $$\underbrace{\int_E [\nabla u_H \cdot n_E]_E\, \varphi_E\, da}_{\text{Scales with } h^{d-1}} = \int_{\omega_E} \nabla u_H \cdot \nabla \varphi_E\, dx$$

$$= \int_{\omega_E} \nabla (u_H - u_h)\, \nabla \varphi_E\, dx + f_h \varphi_E\, dx$$

$$\le |u_H - u_h|_{1,2,\omega_E}\, \underbrace{\|\nabla \varphi_E\|_{0,2,\omega_E}}_{\le C_4 h^{\frac{d}{2}-1}} + \|f_h\|_{0,2,\omega_E}\, \underbrace{\|\varphi_E\|_{0,2,\omega_E}}_{\le C_5 h^{\frac{d}{2}}}$$

Now we square this estimate (& : h^{d-2}):

$$C_6\, h(E)\, \|[\nabla u_H \cdot n_E]_E\|^2_{0,2,E} \underset{(a+b)^2 \le 2a^2 + 2b^2}{\le} 2\, C_4^2\, |u_H - u_h|^2_{1,2,\omega_E} + 2\, C_5^2\, \|h f_h\|^2_{0,2,\omega_E}$$

$$\Rightarrow \sum_{E\in\mathcal{E}_{ref}} h(E)\, \|[\nabla u_H \cdot n_E]_E\|^2_{0,2,E}$$

$$\le \frac{1}{C_6} \sum_{E\in\mathcal{E}_{ref}} \left(2 C_4^2\, |u_H - u_h|^2_{1,2,\omega_E} + 2 C_5^2\, \|h f_h\|^2_{0,2,\omega_E} \right)$$

$$\underset{\substack{M=2 \\ \text{for } d=2}}{\le} \underbrace{\frac{2 M C_4^2}{C_6}}_{=: C_2} |u_H - u_h|^2_{1,2,\Omega} + \underbrace{\frac{2 M C_5^2}{C_6}}_{=: C_3} \|h f_h\|^2_{0,2,\Omega}$$

(ii) replace in (i) $u_{el}\ f_e$ by u, f and use edge bubble as in 2.4. $\Rightarrow$ 2nd estimate.

$\square$

2.12 Thm (convergence of an adaptive Method)

Assumptions as above, in particular (F) is fulfilled for sufficiently small μ and a fixed $\epsilon > 0$, then based on a marking strategy for simplices, which refined a subset $\mathcal{J}_H^* \subset \mathcal{J}_H$ on each grid level with grid size H, such that

(R) $$\eta^s_{\mathcal{J}_H^*} := \sqrt{\sum_{T \in \mathcal{J}_H^*} (\eta^s_T)^2} \geq (1-\Theta) \sqrt{\sum_{T \in \mathcal{J}_H} (\eta^s_T)^2} =: \eta^s_{\mathcal{J}_H}$$

with $$\eta^s_T := \sum_{E \in \mathcal{E}^\circ(T)} h(E) \, \| [\nabla u_H \cdot n_E]_E \|^2_{0,2,E}$$

(here T refined means all edges of T are refined!),

then for the refined triangulation $\mathcal{J}_h$ we obtain

$$|e_h|_{1,2,\Omega} \leq K \, |e_H|_{1,2,\Omega}$$

for a $K < 1$ or $|e_H|_{1,2,\Omega} \leq \epsilon$.

Pf $$|u_h - u_H|^2_{1,2} \overset{2.11.}{\geq} \frac{1}{C_2} \underbrace{\sum_{E \in \mathcal{E}^\circ(\mathcal{J}_H^*)} h_E \| [\nabla u_H \cdot n_E]_E \|^2_{0,2,E}}_{\geq \left(\eta^s_{\mathcal{J}_H^*}\right)^2 \geq (1-\Theta)^2 \left(\eta^s_{\mathcal{J}_H}\right)^2} - \frac{C_3}{C_2} \underbrace{\| h f_h \|^2_{0,2,\Omega}}_{\leq \mu^2 \epsilon^2}$$

$$\geq \frac{1}{C_2} (1-\Theta)^2 \left(\eta^s_{\mathcal{J}_H}\right)^2 - \frac{C_3}{C_2} \mu^2 \epsilon^2$$

recall robustness estimate (2.3):

$$\left[\begin{aligned} |e_H|_{1,2,\Omega} \le{}& C_7\, \eta^S_{J_H} + C_8 \|H f_H\|_{0,2,\Omega} + C_p \|f - f_H\|_{0,2,\Omega} \\ &+ C_9 \,|u^\partial - u^\partial_H|_{1,2,\Omega} \end{aligned}\right]$$

$$\Rightarrow \eta^S_{J_H} \ge \frac{1}{C_7}\Big(|e_H|_{1,2,\Omega} - C_8 \underbrace{\|H f_H\|_{0,2,\Omega}}_{\le \mu\epsilon} - C_p \underbrace{\|f - f_H\|_{0,2,\Omega}}_{\mu\epsilon} - C_9 \underbrace{|u^\partial - u^\partial_H|_{1,2,\Omega}}_{\le \mu\epsilon}\Big)$$

$$\ge \underbrace{\frac{1}{C_7}|e_H|_{1,2,\Omega}}_{=e} - \underbrace{\overbrace{\frac{C_8 + C_p + C_9}{C_7}}^{=:C_{10}}\mu\epsilon}_{b}$$

$$\Rightarrow |u_e - u_H|^2_{1,2} \ge \frac{(1-\Theta)^2}{2 C_2 C_7^2} |e_H|^2_{1,2,\Omega} - \left(C_{10}^2 + \frac{C_3}{C_2}\right)\mu^2\epsilon^2$$

$$\boxed{\begin{aligned} &\eta > e - b \\ \Rightarrow\ &\eta^2 > \tfrac{1}{2}e^2 - b^2 \end{aligned}}$$

$$\boxed{|e_H|_{1,2,\Omega} \ge \epsilon} \qquad \ge \Bigg(\underbrace{\frac{(1-\Theta)^2}{2 C_2 C_7}}_{=: C_{11}(1-\Theta)^2} - \underbrace{\left(C_{10}^2 + \frac{C_3}{C_2}\right)}_{=: C_{12}}\mu^2 \Bigg) |e_H|^2_{1,2,\Omega}$$

$$\xRightarrow{2.10.} |e_H|^2_{1,2,\Omega} - |e_e|^2_{1,2,\Omega} \ge \underbrace{\tfrac{1}{2}|u_e - u_H|^2_{1,2,\Omega}}\; - C_1\left(C_H \mu + (C_H \mu)^2\right)|e_H|^2$$

$$\ge \left(C_{11}(1-\Theta)^2 - C_{12}\mu^2\right)|e_H|^2_{1,2}$$

$$\Rightarrow |e_e|^2_{1,2,\Omega} \le \underbrace{\left[1 - C_{11}(1-\Theta)^2 + C_{12}\mu^2 + C_1\left(C_H \mu + (C_H\mu)^2\right)\right]}_{K^2(\Theta,\mu) \ge 0} \cdot |e_H|^2$$

choose μ small enough such that $K(\Theta, \mu) < 1$ $\square$

Remark: How to fulfil the requirement R ?

(i) sorting of $(\eta_T^S)_{T\in J_H}$ and choose top down a set of simplices to be marked! $\rightsquigarrow O(\ln(\# J_H)\, \# J_H)$ cost!

(ii) Sort $(\eta_T^S)_{T\in J_H}$ in M baskets

$$\left[m \max_T \eta_T^S, (m+1) \max_T \eta_T^S\right) \quad \text{for } m = 0, \ldots M-1$$

and refine top down sufficiently many baskets

$\rightsquigarrow O(\# J_H)$ cost!

3. Non conforming FE

Abstract error estimates:

$(\tilde{P})$ $\boxed{\text{continuous pb}}$ Find $u \in V$: $a(u,v) = \ell(v) \quad \forall v \in V$

$(\tilde{P}_h)$ $\boxed{\text{discrete pb}}$ Find $u_h \in V_h$: $a_h(u_h, v_h) = \ell_h(v_h) \quad \forall v_h \in V^h$

assumption: a_h uniformly elliptic, i.e.

$$a_h(v,v) \geq \alpha \|v\|^2$$

important: a_h is still defined on $V \times V$ and $V^h \subset V$

3.1. **Lemma** (Strang 1st) Assume a bd, a_h uniformly elliptic on $V^h \times V^h$, $V^h \subset V$, th

$$\|u - u_h\| \leq C\left(\inf_{v_h \in V^h} \left\{ \|u - v_h\| + \sup_{w_h \in V^h} \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} \right\} + \sup_{w_h \in V^h} \frac{|\ell(w_h) - \ell_h(w_h)|}{\|w_h\|} \right)$$

Pf (see also exercise)

$$\alpha \|u_h - v_h\|^2 \leq a_h(u_h - v_h, \underbrace{u_h - v_h}_{= w_h})$$

$$= a(u - v_h, w_h) + [a(v_h, w_h) - a_h(v_h, w_h)]$$

$$+ \underbrace{[a_h(u_h, w_h) - a(u, w_h)]}_{= \ell_h(w_h) - \ell(w_h)}$$

$$\Rightarrow \|u_h - v_h\| \leq C\left(\|u - v_h\| + \frac{|a(v_h, w_h) - a_h(v_h, w_h)|}{\|w_h\|} + \frac{|\ell_h(w_h) - \ell(w_h)|}{\|w_h\|} \right)$$

$$\|u - u_h\| \leq \|u - v_h\| + \|u_h - v_h\|$$

$\Rightarrow$ claim $\square$

now: we do not require $V^h \subset V$!

We denote by $\|\cdot\|_h$ a norm on $V^h \oplus V$

3.2. **Lemma** (Strang 2^{nd}) Assume $a_h : V^h \oplus V \times V^h \oplus V$,
$a_h(v_h, v_h) \geq \alpha \|v_h\|_h^2$ (uniformly elliptic on V^h) and
$a_h(u, v) \leq C_1 \|u\|_h \|v\|_h \quad \forall u, v \in V^h \oplus V$, then

$$\|u - u_h\|_h \leq C_1 \Big(\underbrace{\inf_{v_h \in V^h} \|u - v_h\|_h}_{\text{approximation error}} + \underbrace{\sup_{w_h \in V^h} \frac{|a_h(u, w_h) - \ell_h(w_h)|}{\|w_h\|_h}}_{\text{consistency error}} \Big)$$

Pf
$$\alpha \|u_h - v_h\|_h^2 \leq a_h(u_h - v_h, \underbrace{u_h - v_h}_{= w_h})$$
$$\overset{(\hat{P}_h)}{=} a_h(u - v_h, w_h) + [\ell_h(w_h) - a_h(u, w_h)]$$

$$\left.\begin{aligned} \Rightarrow \|u_h - v_h\|_h &\leq \frac{C_1}{\alpha} \|u - v_h\|_h + \frac{|a_h(u, w_h) - \ell_h(w_h)|}{\|w_h\|_h} \\ \|u - u_h\|_h &\leq \|u - v_h\|_h + \|u_h - v_h\|_h \end{aligned}\right\} \Rightarrow \text{claim}$$
$\square$

In what follows we also need an adaptation of the Aubin - Nitsche estimate:

3.3. **Lemma** Assumptions as in 3.2. Furthermore, let H be a Hilbert space with scalar product $(\cdot,\cdot)$, V cont. embedded in H, $|\cdot|$ norm on H, $V^h \subset H$, $a(v, w) = a_h(v, w) \quad \forall v, w \in V$, $\ell = \ell_h$, then

$$|u - u_h| \leq \sup_{\substack{w \in H \\ w \neq 0}} \frac{1}{|w|} \Big\{ C_1 \|u - u_h\|_h \|\varphi_w - \varphi_{w,h}\|_h + |a_h(u - u_h, \varphi_w) - (u - u_h, w)| + |a_h(u, \varphi_w - \varphi_{w,h}) - \ell(\varphi_w - \varphi_{w,h})| \Big\}$$

where

φ_w solves $a(\varphi, \varphi_w) = (w, \varphi) \quad \forall \varphi \in V$ and

$\varphi_{w,h}$ solves $a_h(\varphi_h, \varphi_{w,h}) = (w, \varphi_h) \quad \forall \varphi_h \in V^h$.

Pf

$$
\begin{aligned}
(u-u_h, w) &= a_h(u, \varphi_w) - a_h(u_h, \varphi_{w,h}) \\
&= a_h(u-u_h, \varphi_w - \varphi_{w,h}) \\
&\quad + a_h(u_h, \underset{①}{\varphi_w} - \underset{②}{\varphi_{w,h}}) + a_h(\underset{③}{u} - \underset{④}{u_h}, \varphi_{w,h}) \\
&= a_h(u-u_h, \varphi_w - \varphi_{w,h}) \\
&\quad - \left[a_h(\underset{⑤}{u} - \underset{①}{u_h}, \varphi_w) - (a_h(\underset{⑥}{u}, \varphi_w) + a_h(u_h, \underset{②}{\varphi_{w,h}}) \right] \\
&\quad - \left[a_h(\underset{⑥}{u}, \varphi_w - \underset{③}{\varphi_{w,h}}) - \ell(\underset{⑤}{\varphi_w}) + a_h(\underset{④}{u_h}, \varphi_{w,h}) \right] \\
&= a_h(u-u_h, \varphi_w - \varphi_{w,h}) \\
&\quad - \left[a_h(u-u_h, \varphi_w) - (\underset{⑥}{u} - \underset{②}{u_h}, w) \right] \\
&\quad - \left[a_h(u, \varphi_w - \varphi_{w,h}) - \ell(\underset{⑤}{\varphi_w} - \underset{④}{\varphi_{w,h}}) \right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow |u-u_h| &= \sup_{w \neq 0} \frac{(u-u_h, w)}{|w|} \\
&\leq \sup_{w \neq 0} \frac{1}{|w|} \Big(C_1 \|u-u_h\|_h \|\varphi_w - \varphi_{w,h}\|_h + \\
&\qquad |a_h(u-u_h, \varphi_w) - (u-u_h, w)| + \\
&\qquad |a_h(u, \varphi_w - \varphi_{w,h}) - \ell(\varphi_w - \varphi_{w,h})| \Big)
\end{aligned}
$$

$\square$

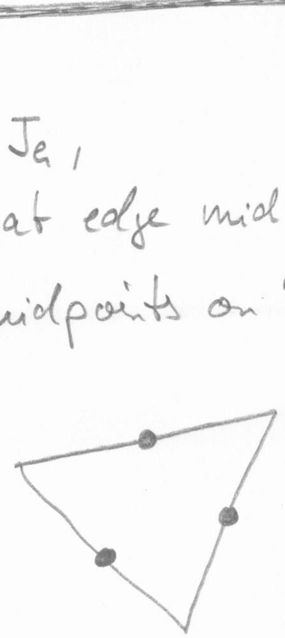
Now we study a nonconforming FE and apply it to Poisson's pb: the **Crouzeix – Raviart – Element**

$$V = H_0^{1,2}(\Omega)$$

$$V_h = \{ v \in L^2(\Omega) \mid v|_T \text{ affine } \forall T \in \mathcal{T}_h, \; v \text{ is continuous at edge midpoints} \}$$

$$V_{h,0} = \{ v \in V_h \mid v = 0 \text{ at edge midpoints on } \partial\Omega \}$$

where $\mathcal{T}_h$ is a regular triangulation of the polygonal domain $\Omega \subset \mathbb{R}^{d=2}$



$$V^h = V_{h,0}$$

<u>discrete problem:</u>

$$a_h(u,v) := \sum_{T\in\mathcal{T}_h} \int_T \nabla u \cdot \nabla v \, dx \quad \text{for } u,v \in H^{1,2}(\Omega) \oplus \mathcal{V}_{h,0}$$

$$\| v \|_h := \sqrt{a_h(v,v)}$$

<u>continuous problem:</u>

$$\begin{aligned} -\Delta u &= f \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \qquad (P)$$

<u>weak formulation:</u> $a_h(u,\varphi) = (f,\varphi) \;\; \forall \varphi \in H^{1,2}_0(\Omega)$

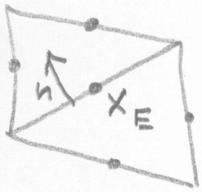
<u>assumption:</u> Ω convex polyeder $\underset{\text{regularity theory}}{\Longrightarrow}$ $u \in H^2(\Omega)$ and $\|u\|_{2,2,\Omega} \le \|f\|_{0,2,\Omega}$

<u>interpolation operator:</u>

$$I_h^{CR} v : H^{2,2}(\Omega) \to \mathcal{V}_h \quad \text{with. } I_h^{CR} v(x_E) = v(x_E)$$

for all edge center x_E

<u>with respect to Lemma 3.2. we compute:</u>



$$L_u(w_h) := a_h(u,w_h) - \ell(w_h)$$

$$= \sum_T \left(\int_T \nabla u \cdot \nabla w_h - f w_h \, dx \right)$$

$\boxed{\partial_n u = \nabla u \cdot n}$

$$= \sum_T \left(\int_{\partial T} \partial_n u \, w_h \, da - \int_T \underbrace{(\Delta u + f)}_{=0} w_h \, dx \right) = \sum_T \int_{\partial T} \partial_n u \, w_h \, da$$

$\boxed{0 = [w_h(x_E)], \;\; 0 = w_h(x_E) \; \forall x_E \in \partial\Omega}$

$$= \sum_T \sum_{E\subset\partial T} \int_E \partial_n u \, (w_h - \overline{w}_{h,E}) \, da ,$$

<u>where:</u> $\overline{w}_{h,E} = \fint_E w_h \, da = w_h(x_E)$

$\boxed{0 = \int_E \beta \, (w_h - w_h(x_E)) \, da \;\; \forall \beta \in \mathbb{R}}$

$\boxed{\beta = \partial_n I_h^{CR} u}$

$$= \sum_T \sum_{E\subset\partial T} \int_E \partial_n (u - I_h^{CR} u)(w_h - \overline{w}_{h,E}) \, da$$

$$\Rightarrow |L_u(w_h)| \le \sum_T \sum_{E\subset\partial T} \|\nabla(u - I_h^{CR} u)\|_{0,2,E} \, \| w_h - \overline{w}_{h,E} \|_{0,2,E}$$

- $\| \nabla (u - I_h^{CR} u) \|_{0,2,E} \overset{\text{scaling}}{\leq} C\, h^{\frac{1}{2}-1} \| \hat{\nabla} (\hat{u} - \hat{I}^{CR} \hat{u}) \|_{0,2,\hat{E}}$

("^" reference element)

trace thm

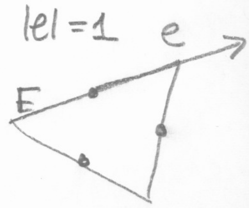
$\leq C\, h^{\frac{1}{2}-1} \| \hat{\nabla} (\hat{u} - \hat{I}^{CR} \hat{u}) \|_{1,2,\hat{T}}$

$\hat{I}^{CR}|_{P_1} = \mathbb{1}$, norm equiv

$\leq C\, h^{-\frac{1}{2}} |\hat{u}|_{2,2,\hat{T}} \overset{\text{scaling}}{\leq} C\, h^{-\frac{1}{2}+2-1} |u|_{2,2,T}$

$\Rightarrow \| \nabla (u - I_h^{CR} u) \|_{0,2,E} \leq C\, h^{\frac{1}{2}} |u|_{2,2,T}$

- $\| w_h - \overline{w}_{h,E} \|_{0,2,E} \overset{\text{scaling}}{\leq} C\, h^{\frac{1}{2}} \| \hat{w}_h - \overline{\hat{w}}_{h,E} \|_{0,2,\hat{E}}$



Poincaré

$\leq C\, h^{\frac{1}{2}} C_P \| \nabla \hat{w}_h \cdot e \|_{0,2,\hat{E}}$

$\circledast$

$\leq C\, h^{\frac{1}{2}} \| \nabla \hat{w}_h \|_{0,2,\hat{T}}$

$\overset{\text{scaling}}{\leq} C\, h^{\frac{1}{2}-1+1} \| \nabla w_h \|_{0,2,T}$

$\Rightarrow \| w - \overline{w}_{h,E} \|_{0,2,E} \leq C\, h^{\frac{1}{2}} |w_h|_{1,2,T}$

---

altogether:

$$|L_u(w_h)| \leq \sum_T 3 C\, h\, |u|_{2,2,T} |w_h|_{1,2,T}$$

$$\leq C\, h\, |u|_{2,2,\Omega} \|w_h\|_h$$

$\mathcal{V}_{h,0}^{conf} \subset \mathcal{V}_{h,0} \Longrightarrow \inf_{v_h \in \mathcal{V}_{h,0}} \|u - v_h\|_h \leq C\, h\, |u|_{2,2,\Omega}$

conforming, p.w. affine FE space

3.2.

$$\Longrightarrow \boxed{\|u - u_h\|_h \leq C\, h\, |u|_{2,2,\Omega} \leq C\, h\, \|f\|_{0,2,\Omega}}$$

Next we apply the generalized Aubin-Nitsche-Lemma 3.3 to the Crouzeix-Raviart-Element $\longrightarrow$

$$\left.\begin{array}{l} a(\varphi, \varphi_w) = (w, \varphi)\ \forall \varphi \in H_0^{1,2}(\Omega) \\ a_h(\varphi_h, \varphi_{w,h}) = (w, \varphi_h)\ \forall \varphi_h \in V_{h,0} \end{array}\right\} \Rightarrow \varphi_w - \varphi_{w,h} \text{ is a}$$

discretization error as $u - u_h$

$$\Rightarrow \|\varphi_w - \varphi_{w,h}\|_h \le C_1 h\, |\varphi_w|_{2,2,\Omega} \le C_1 h\, \|w\|_{0,2,\Omega}$$

Furthermore the above estimate for $L_u(\cdot)$ holds also for $w \in V_{h,0} \oplus H_0^{1,2}(\Omega)$ due to the trace theorem ⊛:

$$\|\hat{v}\|_{1,2,\hat{T}} \quad \text{and} \quad \left|\int_{\hat{E}} \hat{v}\, d\hat{a}\right| + \|\hat{\nabla}\hat{v}\|_{0,2,\hat{T}} \quad \text{are equiv. norms}$$

(**)

$$\Rightarrow \|\hat{w} - \overline{\hat{w}_{\hat{E}}}\|_{0,2,\hat{E}} \le C_1 \|\hat{\nabla}\hat{w}\|_{0,2,\hat{T}}$$

Thus,

$$\begin{aligned} |a_h(u-u_h, \varphi_w) - (u-u_h, w)| &= |L_{\varphi_w}(u-u_h)| \\ &\le C_1 h\, |\varphi_w|_{2,2,\Omega} \|u-u_h\|_h \\ &\le C_1 h\, |w|_{0,2,\Omega} \|u-u_h\|_h \end{aligned}$$

$$\begin{aligned} |a_h(u, \varphi_w - \varphi_{w,h}) - (f, \varphi_w - \varphi_{w,h})| &= |L_u(\varphi_w - \varphi_{w,h})| \\ &\le C_1 h\, \underbrace{|u|_{2,2,\Omega}}_{\le C_1 \|f\|_{0,2,\Omega}} \|\varphi_w - \varphi_{w,h}\|_h \end{aligned}$$

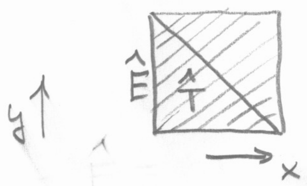
Finally we get by Lemma 3.3.:

$$\begin{aligned} \|u-u_h\|_{0,2,\Omega} &\le C_1 h \|u-u_h\|_h + C_1 h \|u-u_h\| \\ &\quad + C_1 h^2 \|f\|_{0,2,\Omega} \\ &\le C_1 h^2 \|f\|_{0,2,\Omega} \end{aligned}$$

$$\boxed{\begin{array}{l}\|\varphi_w - \varphi_{w,h}\|_h \\ \le C_1 h \|w\|_{0,2,\Omega}\end{array}}$$

$$\boxed{\|u-u_h\|_h \le C_1 h \|f\|_{0,2}}$$

ad ✱✱ :



at first we extend $\hat{v} : \hat{T} \to \mathbb{R}$ to $D = [0,1]^2$ via

$$\hat{v}(1-y, 1-x) := \hat{v}(x,y)$$

the claim follows if we show that

$\|\hat{v}\|_{1,2,[0,1]^2}$ and $|\hat{v}_E| + \|\nabla \hat{v}\|_{0,2,[0,1]^2}$ are equivalent norms

with $\hat{v}_E = \int_0^1 \hat{v}(0,y)\,dy$

For $\overline{\hat{v}}_D = \int_0^1\int_0^1 \hat{v}(x,y)\,dx\,dy$:

$$\|\hat{v}\|_{0,2,D} \leq \underbrace{\|\hat{v} - \hat{v}_D\|_{0,2,D}} + \underbrace{\|\hat{v}_D\|_{0,2,D}}_{= |\hat{v}_D|}$$

$$\leq C_P \|\nabla \hat{v}\|_{0,2,D}$$

Furthermore:

$$|\hat{v}_D| = \left| \int_0^1\int_0^1 \hat{v}(x,y)\,dy\,dx \right|$$

$$= \left| \int_0^1\int_0^1 \hat{v}(0,y) + \int_0^x \partial_x \hat{v}(\xi,y)\,d\xi \; dy\,dx \right|$$

$$\leq |\hat{v}_E| + \|\nabla \hat{v}\|_{0,2,D}$$

cf Poincaré

$$\Rightarrow \boxed{\|\hat{v}\|_{0,2,D} \leq |\hat{v}_E| + \|\nabla \hat{v}\|_{0,2,D}}$$

The estimate $|\hat{v}_E| \leq C \|\hat{v}\|_{1,2,D}$ follows by the trace thm

Furthermore $\|\hat{v}\|^2_{1,2,D} = 2\|\hat{v}\|^2_{1,2,T}$

$\Rightarrow$ claim

□

to summarize:

3.4. Thm $\boxed{\text{ }}$

Ω convex, polygonally bd, $\mathcal{T}_h$ regular triangulation, $f \in L^2(\Omega)$, then the error estimate

$$\|u - u_h\|_{0,2,\Omega} + h\,\|u - u_h\|_h \leq C\,h^2\,\|f\|_{0,2,\Omega}$$

holds for the solution u_h of the Crouzeix-Raviart discretization.

4. Saddle point pbs and mixed finite elements

preliminaries: $\mathcal{U}, V$ Hilbertspaces, $a: \mathcal{U} \times V \to \mathbb{R}$ bilinear form
$L: \mathcal{U} \to V'$ defined via $(Lu)(v) = a(u,v)$,
where we suppose $a(\cdot,\cdot)$ to be bd.

$(\tilde{P})$ variational problem: for given $f \in V'$ find $u \in \mathcal{U}$, s.t.
$a(u,v) = \langle f, v \rangle \quad \forall v \in V$
$\Leftrightarrow \quad u = L^{-1} f$

4.1. Thm L is an isomorphism (L^{-1} exists, L, L^{-1} continuous) if

(i) a is continuous $|a(u,v)| \leq C_1 \|u\|_{\mathcal{U}} \|v\|_V$ ($\Leftrightarrow$ bd)

(ii) inf-sup-condition holds:

$$\sup_{v \in V} \frac{a(u,v)}{\|v\|_V} \geq \alpha \|u\|_{\mathcal{U}} \quad \forall u \in \mathcal{U} \text{ with } \alpha > 0$$

$$\Leftrightarrow \inf_{u \in \mathcal{U}} \sup_{v \in V} \frac{a(u,v)}{\|u\|_{\mathcal{U}} \|v\|_V} \geq \alpha$$

(iii) $\forall v \in V \ v \neq 0 \ \exists u \in \mathcal{U}$ s.t. $a(u,v) \neq 0$

Furthermore, $L: \mathcal{U} \to \{ v \in V \mid a(u,v) = 0 \ \forall u \in \mathcal{U} \}^0 \subset V'$
is an isomorphism if (i), (ii) hold.
Here $W^0 := \{ v' \in V' \mid \langle v', w \rangle = 0 \ \forall w \in W \}$ for W closed subset of V
(polar or annihilator of W)

Remark $\mathcal{U} = V \Rightarrow \begin{cases} \text{(ii)} \Leftrightarrow \text{coercivity} \\ \text{(iii)} \Leftarrow \text{(ii)} \\ \text{4.1.} \Leftrightarrow \text{Lax Milgram} \end{cases}$

Pf recall: $A : X \to Y$ linear, then

$A^* : Y' \to X'$ with $\langle A^* y', x \rangle := \langle y', Ax \rangle$

now: $L : \mathcal{U} \to V'$, thus

$L^* : \underset{\substack{\uparrow \\ \text{reflexivity of} \\ \text{Hilbertspaces}}}{V'' = V} \to \mathcal{U}'$ $\quad \langle L^* \underset{\substack{\uparrow \\ \text{in } V''}}{v}, u \rangle = \langle v, Lu \rangle = a(u,v) = \langle Lu, \underset{\substack{\uparrow \\ \text{in } V}}{v} \rangle$

At first (ii) $\Rightarrow$ L injective, indeed

$Lu_1 = Lu_2 \Rightarrow a(u_1 - u_2, v) = 0 \ \forall v \in V \overset{(ii)}{\Rightarrow} \|u_1 - u_2\| = 0$

$\Rightarrow$ for $f \in L(\mathcal{U})$ (range of L) $\exists!\, u = L^{-1} f$ with

$$\alpha \|u\|_{\mathcal{U}} \le \sup_{v \in V} \frac{a(u,v)}{\|v\|} = \sup_{v \in V} \frac{\langle f, v \rangle}{\|v\|} = \|f\|_{V'}$$

$\Leftrightarrow \alpha \|u\|_{\mathcal{U}} \le \|Lu\|$

$\Rightarrow$ L^{-1} continuous on $L(\mathcal{U})$ and $|||L^{-1}||| \le \frac{1}{\alpha}$

Next L, L^{-1} continuous $\Rightarrow$ $L(\mathcal{U})$ closed

$\Big(Lu_j \xrightarrow[j \to \infty]{} \ell \in V' \overset{L^{-1}\text{ continuous}}{\Longrightarrow} u_j = L^{-1} L u_j$ Cauchy sequence

$\Rightarrow u_j \to u \in \mathcal{U} \overset{L \text{ continuous}}{\Longrightarrow} Lu_j \to Lu = \ell \Big)$

closed range thm (see below) $\Rightarrow$

$$L(\mathcal{U}) = (\ker L^*)^0 = \{ v \in V' \mid \underbrace{\langle L^* v, u \rangle}_{= a(u,v)} = 0 \ \forall u \in \mathcal{U} \}$$

$\Rightarrow$ 2nd claim

Finally (iii) $\Rightarrow \{ v \in V \mid a(u,v) = 0 \ \forall u \in \mathcal{U} \} = \{0\}$

$\Rightarrow L(\mathcal{U}) = V' \Rightarrow$ 1st claim $\square$

4.2. **Thm** (Closed range thm)

X, Y Banach spaces, $L: X \to Y$ bounded linear operator then the following statements are equivalent

(i) $L(X)$ is closed in Y

(ii) $L(X) = (\ker L^*)^\circ$

where for $\underset{\text{closed}}{W} \subset Y'$: $W^\circ = \{ y \in Y \mid \langle y', y \rangle = 0 \ \forall_{y' \in W} \}$

Pf it is sufficient to show $\overline{L(X)} = (\ker L^*)^\circ$ ⊛

($\ker L^*$ is closed) $\rightsquigarrow$ ((ii) $\overset{⊛}{\Rightarrow}$ (i)), (i) $\Rightarrow L(X) = \overline{L(X)} \overset{⊛}{=} (\ker L^*)^\circ$

$\boxed{\text{now:}}$

$$(\ker L^*)^\circ = \{ y \in Y \mid \langle y', y \rangle = 0 \ \forall_{y' \in \ker L^*} \}$$

$$= \{ y \in Y \mid \langle y', y \rangle = 0 \ \forall_{y',\ \underbrace{\langle L^* y', x \rangle}_{= \langle y', Lx \rangle} = 0 \ \forall x \in X} \}$$

$\Rightarrow L(X) \subset (\ker L^*)^\circ$

$(\ker L^*)^\circ$ is the intersection of closed subsets $\Rightarrow (\ker L^*)^\circ$ closed

$\Rightarrow \overline{L(X)} \subset (\ker L^*)^\circ$

Now assume $\exists_{y_0 \in (\ker L^*)^\circ}$ with $y_0 \notin \overline{L(X)}$

$\Rightarrow \operatorname{dist}(y_0, L(X)) > 0$

$L(X)$ convex

$\xRightarrow[\text{separation theorem}]{}$ $\exists \, y' \in Y', \alpha \in \mathbb{R} \quad \langle y', y_0 \rangle > \alpha > \langle y', Lx \rangle \ \forall_{x \in X}$

$\xRightarrow{L \text{ linear}}$ $\langle y', Lx \rangle = 0 \ \forall x \in X \Rightarrow \alpha > 0 \Rightarrow \langle y', y_0 \rangle \neq 0$

↯ with the definition of $(\ker L^*)^\circ$ because $y' \in \ker L^*$ $\square$

Now we consider finite dimensional subspaces $\mathcal{U}_h \subset \mathcal{U}$, $\mathcal{V}_h \subset \mathcal{V}$ and solve:

$$\boxed{(\tilde{P}_h) \quad \text{Find } u_h \in \mathcal{U}_h, \text{ s.t. } a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in \mathcal{V}_h}$$

4.3 <u>Thm</u> (existence of discrete solutions, abstract convergence)

Assumptions as above, $|a(u,v)| \leq C_1 \|u\|_{\mathcal{U}} \|v\|_{\mathcal{V}}$,

$$\inf_{u_h \in \mathcal{U}_h} \sup_{v_h \in \mathcal{V}_h} \frac{a(u_h, v_h)}{\|u_h\|_{\mathcal{U}} \|v_h\|_V} \geq \alpha > 0 \quad \forall u_h \in \mathcal{U}_h, v_h \in \mathcal{V}_h,$$

$$\forall v_h \in \mathcal{V}_h, v_h \neq 0 \quad \exists u_h \in \mathcal{U}_h \quad \text{s.t.} \quad a(u_h, v_h) \neq 0$$

then $\exists!$ discrete solution u_h of $(\tilde{P}_h)$ and

$$\|u - u_h\|_{\mathcal{U}} \leq \left(1 + \frac{C_1}{\alpha}\right) \inf_{w_h \in \mathcal{U}_h} \|u - w_h\|_{\mathcal{U}}$$

<u>Pf</u> • existence $\Leftarrow$ 4.1.

• $(\tilde{P}), (\tilde{P}_h) \Rightarrow \quad a(u - u_h, v_h) = 0 \quad \forall v_h \in \mathcal{V}_h$

$$\Rightarrow \quad |a(u_h - w_h, v_h)| = |a(u - w_h, v_h)| \quad \forall v_h \in \mathcal{V}_h$$
$$\leq C_1 \|u - w_h\|_{\mathcal{U}} \|v_h\|_V$$

• $L_h : \mathcal{U}_h \to \mathcal{V}_h'$ defined as

$$\langle L_h u_h, v_h \rangle = a(u_h, v_h) \Rightarrow |||L_h^{-1}||| \leq \frac{1}{\alpha}$$

$$\alpha \|u_h\|_{\mathcal{U}} \leq \sup_{v_h} \frac{a(u_h, v_h)}{\|v_h\|_V} = \sup_{v_h} \frac{\langle L u_h, v_h \rangle}{\|v_h\|_V} \leq \|L u_h\|_{\mathcal{U}}$$

(see above)

$$\Rightarrow \|u_h - w_h\|_{\mathcal{U}} \leq \frac{1}{\alpha} \|L_h(u_h - w_h)\|_{V'}$$
$$= \frac{1}{\alpha} \sup_{v_h \in \mathcal{V}_h} \frac{a(u_h - w_h, v_h)}{\|v_h\|_V} \leq \frac{C_1}{\alpha} \|u - w_h\|_{\mathcal{U}}$$

$$\Rightarrow \|u-u_h\| \le \|u-w_h\| + \|w_h-u_h\|$$
$$\le \left(1+\frac{C_1}{\alpha}\right)\|u-w_h\|$$

$\square$

Now we study Poissons pb in a mixed formulation:

(P) $-\Delta u = f$ in Ω with Ω Lipschitz domain

$u = 0$ on $\partial\Omega$

Define $\sigma = \nabla u$: $(P) \Leftrightarrow \left[\begin{array}{lll} \sigma - \nabla u & = 0 & \text{in } \Omega \\ -\operatorname{div}\sigma & = f & \text{in } \Omega \\ u = 0 & & \text{on } \partial\Omega \end{array}\right]$

weak formulation:

$(\tilde{P})$ $\quad (\sigma, \tau)_{0,2,\Omega} - (\tau, \nabla u)_{0,2,\Omega} = 0$

$\quad -(\sigma, \nabla v)_{0,2,\Omega} = -\langle f, v\rangle$

$\forall \tau \in (L^2(\Omega))^d,\ v \in H_0^{1,2}(\Omega)$

$$\boxed{\begin{array}{ll} \underline{1^{st}\text{ ansatz}}: & X = (L^2(\Omega))^d,\ M = H_0^{1,2}(\Omega) \\ & a(\sigma,\tau) = (\sigma,\tau)_{0,2,\Omega} \\ & b(\tau,v) = -(\tau,\nabla v)_{0,2,\Omega} \end{array}}$$

$\leadsto (\tilde{P}) \rightarrow \left.\begin{array}{ll} a(\sigma,\tau) + b(\tau,u) & = 0 \\ b(\sigma,v) & = -\langle f,v\rangle \end{array}\right\}$ $(\tilde{P}_m)$ mixed formulation

$\left.\begin{array}{ll} A: X \to X' & \langle A\sigma,\tau\rangle = a(\sigma,\tau) \\ B: X \to M' & \langle B\tau,v\rangle = b(\tau,v) \\ B^*: M \to X' & \langle B^*u,\tau\rangle = b(\tau,u) \end{array}\right\}$ $\begin{array}{ll} A\sigma + B^*u & = 0 \\ B\sigma & = -f \end{array}$

conceptually: $\boxed{A \sim \mathbb{1},\ B^* \sim -\nabla,\ B \sim \operatorname{div}}$

$$b(\tau,v) = -\int_\Omega \tau\cdot\nabla v\,dx = \int_\Omega \operatorname{div}\tau\, v\,dx - \int_{\partial\Omega} \tau\cdot n\,\underbrace{v}_{=0}\,da$$

Now we study an alternative approach:

2nd ansatz: $X = H(\operatorname{div},\Omega) = \{\tau \in L^2(\Omega)^d \mid \operatorname{div}\tau \in L^2(\Omega)\}$

with $\|\tau\|_{H(\operatorname{div},\Omega)} := \left(\|\tau\|^2_{0,2,\Omega} + \|\operatorname{div}\tau\|^2_{0,2,\Omega}\right)^{\frac{1}{2}}$

$M = L^2(\Omega)$

remark: $\boxed{X \text{ is a Hilbert space},\ X \not\supseteq H^{1,2}(\Omega) \text{ for } d>1}$

$$\langle A\sigma,\tau\rangle = a(\sigma,\tau) := (\sigma,\tau)_{0,2,\Omega}$$

$$\boxed{\langle B\tau,v\rangle = b(\tau,v) := (\operatorname{div}\tau,v)} = -(\tau,\nabla v)$$

$\uparrow v\in H^{1,2}_0(\Omega)$

with $A: X\to X'$ $\quad A\sigma + B^*u = 0$

$B: X \to M'$ $\quad B\sigma = -f$

$B^*: M \to X'$

$$a(\sigma,\tau) + b(\tau,u) = 0 \quad \forall \tau\in X = H(\operatorname{div},\Omega)$$

$$b(\sigma,v) = -f \quad \forall v \in M = L^2(\Omega)$$

$\Leftrightarrow \int_\Omega \sigma\tau + \operatorname{div}\tau\, u\,dx = 0 \xrightarrow{u\in H^{1,2}_0(\Omega)} \int_\Omega \sigma\tau - \tau\cdot\nabla u\,dx = 0$

$\int_\Omega \operatorname{div}\sigma\, v\,dx = -\int_\Omega f v\,dx \xrightarrow[v\in H^{1,2}_0(\Omega)]{} \int_\Omega \sigma\nabla v - f v\,dx = 0$

$\Rightarrow (\tilde{P})$ $\boxed{\int_\Omega \nabla u\nabla v - f v\,dx = 0 \quad \forall v\in H^{1,2}_0(\Omega)}$

general concept: $\mathbb{X}$, M Hilbertspaces

$a: \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ symmetric, continuous

$b: \mathbb{X} \times M \to \mathbb{R}$ continuous, $\ell \in \mathbb{X}'$, $g \in M'$

(P_c) constraint minimization pb:

$$J(u) = \frac{1}{2} a(u,u) - \langle \ell, u \rangle \xrightarrow{u \in \mathbb{X}} \min$$

subject to $b(u,\mu) = \langle g, \mu \rangle \quad \forall \mu \in M$

for Poisson's pb: $J(z) = \int_\Omega \frac{z^2}{2} \, dx$

constraint: $-\operatorname{div} z = f$ weakly:

$$b(z,v) = \int_\Omega \operatorname{div} z \, v \, dx = \int_\Omega -f v \, dx \quad \forall v \in L^2(\Omega)$$

Lagrangian: $\mathcal{L}(\underset{\mathbb{X}}{\overset{\uparrow}{u}}, \underset{M}{\overset{\uparrow}{\lambda}}) := J(u) + [\, b(u,\lambda) - \langle g, \lambda \rangle \,]$

saddle point (u,λ) of $\mathcal{L}$:

(P_s)

$$\partial_u \mathcal{L}(u,\lambda)(v) = a(u,v) + b(v,\lambda) - \langle \ell, v \rangle = 0 \quad \forall v \in \mathbb{X}$$

$$\partial_\lambda \mathcal{L}(u,\lambda)(\mu) = b(u,\mu) - \langle g, \mu \rangle = 0 \quad \forall \mu \in M$$

with

$A: \mathbb{X} \to \mathbb{X}' \quad \langle Au, v \rangle = a(u,v)$

$B: \mathbb{X} \to M' \quad \langle Bu, \mu \rangle = b(u,\mu)$

$B^*: M \to X' \quad \langle B^* \lambda, v \rangle = b(v,\lambda)$

we obtain

$$(P_s) \Leftrightarrow \boxed{\begin{aligned} Au + B^*\lambda &= \ell \\ Bu &= g \end{aligned}}$$

with $L : X \times M \to X' \times M'$

$$\langle L(u,\lambda),(v,\mu)\rangle = a(u,v) + b(v,\lambda) + b(u,\mu)$$
$$= \langle \ell, v\rangle + \langle g, \mu\rangle$$

$(P_S) \iff L(u,\lambda) = (\ell, g)$

4.4. Lemma

$$\inf_{\mu\in M} \sup_{v\in X} \frac{b(v,\mu)}{\|v\|_X \|\mu\|_M} \geq \beta > 0$$

$\iff$ $B^* : M \to V^0$ for $V = \overbrace{\{ v\in X \mid b(v,\mu) = 0 \ \forall \mu\in M\}}^{\ker B}$ is an isomorphism and $\|B^*\mu\|_X \geq \beta \|\mu\|_M$

$\iff$ $B : V^\perp \to M'$ is an isomorphism with $\|Bv\|_{M'} \geq \beta \|v\|_X$

Pf ① $\iff$ ② follows from 4.1.

② $\Rightarrow$ for fixed $v \in V^\perp$ $\exists!\, \lambda\in M$ $B^*\lambda = (v,\cdot)_X$

$V^\perp \cong V^0$ Riesz. Reps. Thm $\qquad \Updownarrow$

$$b(w,\lambda) = (v,w)_X \ \forall w\in X$$

$\|B^*\mu\|_{X'} \geq \beta\|\mu\|_M \Rightarrow$

$$\|v\|_X = \|b(\cdot,\lambda)\|_{X'} = \|B^*\lambda\|_{X'} \geq \beta\|\lambda\|_M$$

$w = v$

$$\Rightarrow \sup_{\mu\in M}\frac{b(v,\mu)}{\|\mu\|_M} \geq \frac{b(v,\lambda)}{\|\lambda\|_M} = \frac{(v,v)}{\|\lambda\|_M} = \frac{\|v\|_X^2}{\|\lambda\|_M} \geq \beta\|v\|_X$$

$\Rightarrow$ $B : V^\perp \to M'$ fulfills assumption of 4.1. $\Rightarrow$ ③

③ $\Rightarrow$
$$\|\mu\|_M = \sup_{g\in M'}\frac{\langle g,\mu\rangle}{\|g\|_{M'}} = \sup_{v\in V^\perp}\frac{\langle Bv,\mu\rangle}{\|Bv\|_{M'}}$$
$$= \sup_{v\in V^\perp}\frac{b(v,\mu)}{\|Bv\|_{M'}} \leq \sup_{v\in V^\perp}\frac{b(v,\mu)}{\beta\|v\|_X} \leq \frac{1}{\beta}\sup_{v\in V}\frac{b(v,\mu)}{\|v\|}$$

$\Rightarrow$ ①

$\square$

Now we study the invertibility of the operator

$$L : X \times M \to X' \times M' \quad \langle L(u,\lambda),(v,\mu)\rangle = a(u,v) + b(v,\lambda) + b(u,\mu)$$

or in shorthand notation

$$L = \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix}, \quad \langle L(u,v),(v,\mu)\rangle = \langle L\begin{pmatrix} u \\ \lambda \end{pmatrix}, \begin{pmatrix} v \\ \mu \end{pmatrix}\rangle$$

4.5. **Thm** (L isomorphism)

L is an isomorphism, if

(i) a is V elliptic, i.e. $a(v,v) \geq \alpha \|v\|_X^2$ $\forall v \in V = \{ v \in X \mid b(v,\mu) = 0 \ \forall \mu \in M \}$

(ii) b fulfills the inf-sup-condition.

Pf
- $V(g) = \{ v \in X \mid b(v,\mu) = \langle g,\mu\rangle \ \forall \mu \in M \}$ for $g \in M'$

4.4. $\Rightarrow$ $B : V^\perp \to M'$ isomorphism $\Rightarrow$ $\left.\begin{array}{l} \exists!\, u_g \in V^\perp \ B u_g = g \\ \|u_g\|_X \leq \frac{1}{\beta}\|g\|_{M'} \end{array}\right\} \Rightarrow V(g) \neq \emptyset$

- $w = u - u_g \rightsquigarrow$

$$(P_s) \quad \begin{aligned} a(w,v) + b(v,\lambda) &= \langle \ell, v\rangle - a(u_g, v) \\ b(w,\mu) &= 0 \end{aligned}$$

a V-elliptic $\Rightarrow$ $F(v) := \frac{1}{2} a(v,v) - \langle \ell, v\rangle + a(u_g, v)$ attains its minimum over V for some $w \in V$ with necessary condition

closed subspace

$$\circledast \quad a(w,v) = \langle \ell, v\rangle - a(u_g, v) \quad \forall v \in V$$

- Thus (P_s) is fulfilled if

$$\exists \lambda \in M \quad \text{with} \quad b(v,\lambda) = \underbrace{\langle \ell, v\rangle - a(\overbrace{u_g + w}^{=u}, v\rangle}_{=:\langle h, v\rangle} \ \forall v \in X$$

$$(*) \Rightarrow h \in V^\circ \overset{4.4.}{\Longrightarrow} \exists \lambda \in M \quad b(v,\lambda) = \langle h, v \rangle \ \forall v \in X$$

$$\text{and } \|\lambda\|_M \le \frac{1}{\beta}\|h\|_{X'} \le \frac{1}{\beta}\left(\|e\|_{X'} + C_1\|u\|_X\right)$$

Furthermore:

$$\|u\|_X \le \|u_g\|_X + \|w\|_X \le \frac{1}{\beta}\|g\|_{M'} + \frac{1}{\alpha}\left(\|e\|_{X'} + \frac{C_1}{\beta}\|g\|_{M'}\right)$$

Hence L^{-1} bd $\Rightarrow$ L isomorphism $\square$

- Now we apply the so called mixed FE approach:

$X_h \subset X$, $M_h \in M$ and find $(u_h, \lambda_h) \in X_h \times M_h$, s.t.

$(P_{S,h})$

$$\boxed{\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= \langle f, v_h \rangle \quad \forall v_h \in X_h \\ b(u_h, \mu_h) &= \langle g, \mu_h \rangle \quad \forall \mu_h \in M_h \end{aligned}}$$

4.6 Thm (existence of discrete solutions, abstract convergence estimates)

Assumptions as in 4.5., furthermore

$(i)_h$ a is V_h-elliptic for $V_h = \{ v_h \in X_h \mid b(v_h, \mu_h) = 0 \ \forall \mu_h \in M_h \}$

$\left(a(v_h, v_h) \ge \alpha \|v_h\|_X^2 \right)$

$(ii)_h$ $$\inf_{\mu_h \in M_h} \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X \|\mu_h\|_M} \ge \beta_h > 0$$

Then, $\exists!$ discrete solution $(u_h, \lambda_h) \in X_h \times M_h$ of $(P_{S,h})$

we have:

$$\|u - u_h\|_X + \|\lambda - \lambda_h\|_M \le C\left(\inf_{w_h \in X_h} \|u - w_h\|_X + \inf_{\mu_h \in M_h} \|\lambda - \mu_h\|_M \right)$$

Pf. recall $V(g) = \{ v \in X \mid b(v,\mu) = \langle g,\mu\rangle \ \forall_{\mu\in M} \}$

$V_h(g) := \{ v_h \in X_h \mid b(v_h,\mu_h) = \langle g,\mu_h\rangle \ \forall_{\mu_h\in M_h} \}$

discrete inf-sup-condition $\Rightarrow B_h : V_h^\perp \to M_h'$ Isomorphism

$(V_h = V_h(0))$ $\Rightarrow V_h(g) \neq \emptyset$

$a(\cdot,\cdot)$ coercive, $X_h \subset X$ $\overset{\text{Lax-Milgram}}{\Rightarrow} \exists!\, u_h \in V_h(g)$, such that

$a(u_h, v_h) = \langle f, v_h\rangle \ \forall v_h \in V_h$

$(u_h = \overset{\in V_h}{v_h} + z_h$ with $z_h \in V_h(g))$

- $z_h \in V_h(g)$ arbitrary, $v_h = u_h - z_h \in V_h$

$$a(v_h,v_h) = \underbrace{\langle f, v_h\rangle}_{a(u,v_h) + \underbrace{b(v_h,\lambda)}_{= b(v_h,\lambda-\mu_h)\ (b(v_h,\lambda_h)=0)}} - a(z_h,v_h)$$

$$\Rightarrow \boxed{a(v_h,v_h) = a(u-z_h, v_h) + b(v_h, \lambda - \mu_h)}$$

- $\overset{\text{coercivity, bdness}}{\Rightarrow}$ $$\|v_h\|_X \le \frac{C}{c}\left(\|u-z_h\|_X + \|\lambda-\mu_h\|_M\right)$$

$$\|u-u_h\|_X \le \|u-z_h\|_X + \|u_h - z_h\|_X$$

$$\Rightarrow \|u-u_h\|_X \le C\left(\inf_{z_h\in V_h(g)} \|u-z_h\| + \inf_{\lambda_h\in M_h} \|\lambda-\mu_h\|\right)$$

- now: $$\boxed{\inf_{z_h\in V_h(g)} \|u-z_h\|_X \overset{!}{\le} \left(1+\frac{C}{\beta}\right) \inf_{v_h\in X_h} \|u-v_h\|_X}$$

to this end: $v_h \in X_h$ arbitrary $\Rightarrow \exists!\, w_h \in V_h^\perp$

$$B_h w_h = B_h(u - v_h)$$

with $\|w_h\|_X \leq \frac{1}{\beta} \|B_h(u - v_h)\|_{M_h'} \leq \frac{C_1}{\beta} \|u - v_h\|_X$

now define $z_h = w_h + v_h$:

$$b(z_h, \mu_h) = b(u - v_h, \mu_h) + b(v_h, \mu_h) = b(u, \mu_h) = \langle g, \mu_h \rangle \quad \forall \mu_h \in M_h$$

$\Rightarrow z_h \in V_h(g)$ and

$$\boxed{\|u - z_h\|_X \leq \|u - v_h\|_X + \|w_h\|_X \leq \left(1 + \frac{C_1}{\beta}\right) \|u - v_h\|_X}$$

- finally $\|\lambda - \lambda_h\|_M$:

$$a(u, v_h) + b(v_h, \lambda) = \langle f, v_h \rangle$$

$$\ominus \; a(u_h, v_h) + b(v_h, \lambda_h) = \langle f, v_h \rangle$$

$$\boxed{b(v_h, \lambda_h - \mu_h) = a(u - u_h, v_h) + b(v_h, \lambda - \mu_h) \quad \forall \mu_h \in M_h, \; v_h \in X_h}$$

$$\Rightarrow \|\lambda_h - \mu_h\|_M \underset{(ii)_h}{\leq} \frac{1}{\beta} \sup_{v_h \in X_h} \frac{1}{\|v_h\|_X} \underbrace{b(v_h, \lambda_h - \mu_h)}_{a(u - u_h, v_h) + b(v_h, \lambda - \mu_h)}$$

$$\leq \frac{C_1}{\beta} \left(\|u - u_h\|_X + \|\lambda - \mu_h\|_M \right)$$

$$\Rightarrow \|\lambda - \lambda_h\|_M \leq \|\lambda - \mu_h\|_M + \|\mu_h - \lambda_h\|_M$$

$$\leq \left(1 + \frac{C_1}{\beta}\right) \|\lambda - \mu_h\|_M + \frac{C_1}{\beta} \|u - u_h\|_X$$

$$\underset{\text{see above}}{\leq} C \left(\|\lambda - \mu_h\|_M + \|u - v_h\|_X \right)$$

Q: How to check the discrete inf-sup condition? □

4.7. **Thm** (Fortin)

Assumptions as in 4.5., $X_h \subset X$, $M_h \subset M$, there exists a constant C_F such that for each $v \in X$ there exist a $\Pi_h(v) \in X_h$ with $b(v, \mu_h) = b(\Pi_h(v), \mu_h)\ \forall \mu_h \in M_h$

and $\|\Pi_h(v)\|_X \leq C_F \|v\|_X$,

then the discrete lim inf condition $(4.6.(ii)_h)$ holds.

Pf

$$\mu_h \in M_h : \sup_{v_h \in X_h} \frac{b(v_h, \mu_h)}{\|v_h\|_X} \geq \sup_{v \in X} \frac{b(\Pi_h(v), \mu_h)}{\|\Pi_h(v)\|}$$

$$= \sup_{v \in X} \frac{b(v, \mu_h)}{\|\Pi_h(v)\|_X} \geq \sup_{v \in X} \frac{b(v, \mu_h)}{C_F \|v\|_X}$$

$$\geq \frac{\beta}{C_F} \|\mu_h\|_M \qquad \square$$

Now we verify the inf-sup-condition and the V-ellipt. for the mixed formulation of Poisson pb:

$$a(\sigma, \tau) = \int_\Omega \sigma\tau\, dx, \quad b(\tau, v) = \int_\Omega \operatorname{div}\tau\, v\, dx$$

$$V = \{\tau \in H(\operatorname{div}, \Omega) \mid \operatorname{div}\tau = 0\}$$

$$a(\tau, \tau) = \|\tau\|_{0,2}^2 \underset{\tau \in V}{=} \|\tau\|_{H(\operatorname{div}, \Omega)}^2 \Rightarrow \boxed{a(\cdot,\cdot)\ V\text{ elliptic}}$$

$p \in L^2(\Omega)$, then $\exists\, \tilde{p} \in C_0^\infty(\Omega)$ $\|p - \tilde{p}\|_{0,2,\Omega} \leq \frac{1}{2}\|p\|_{0,2,\Omega}$

$$\tau(x) = \left(\int_{-\infty}^{x_1} \tilde{p}(t, x_{2-d})\, dt, 0, \text{—} 0\right) \Rightarrow$$

$$\boxed{\operatorname{div}\tau = \tilde{p}}$$

$$\int_\Omega |\tau|^2 dx = \int_\Omega \left(\int_{-\infty}^{x_1} \tilde{p}(t, x_{2-d})\, dt \right)^2 dx$$

$$\leq \operatorname{diam}(\Omega) \int_\Omega \int_{\mathbb{R}} \tilde{p}(t, x_{2-d})^2 dt\, dx \leq \operatorname{diam}(\Omega)^2 \, \|\tilde{p}\|^2_{0,2,\Omega}$$

$$\Rightarrow \quad \|\tau\|^2_{H(\operatorname{div},\Omega)} \leq (1 + \operatorname{diam}(\Omega)^2)\, \|\tilde{p}\|^2_{0,2,\Omega}$$

$$\Rightarrow \quad \frac{b(\tau, p)}{\|\tau\|_{H(\operatorname{div},\Omega)}} \geq \frac{(\tilde{p}, p)}{C' \|\tilde{p}\|_{0,2,\Omega}} \geq C \|p\|_{0,2,\Omega}$$

$$\|p\|_{0,2} \geq \|\tilde{p}\|_{0,2} - \|p - \tilde{p}\|_{0,2} \geq \|\tilde{p}\|_{0,2} - \frac{\|p\|_{0,2}}{2}$$

$$\Rightarrow \frac{3}{2} \|p\| \geq \|\tilde{p}\|$$

$$(p, p)_{0,2} = \underbrace{(p - \tilde{p}, p)_{0,2}}_{\leq \frac{1}{2}\|p\|^2_{0,2}} + (\tilde{p}, p)$$

$$\Rightarrow \frac{1}{2} \|p\|^2 \leq (\tilde{p}, p)$$

The Raviart-Thomas element $(d = 2)$:

$$\Sigma_h = \left\{ \tau_h \in L^2(\Omega)^2 \;\middle|\; \tau_h|_T(x) = \begin{pmatrix} a_T \\ b_T \end{pmatrix} + c_T x \text{ with } a_T, b_T, c_T \in \mathbb{R} \;\; \forall T \in \mathcal{J}_h, \;\; \tau_h \cdot n \text{ continuous on edges} \right\}$$

$$M_h = \{ p_h \in L^2(\Omega) \mid p_h|_T = d_T \text{ with } d_T \in \mathbb{R} \;\; \forall T \in \mathcal{J}_h \}$$

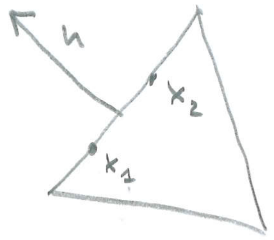
where $\mathcal{J}_h$ is a regular, admissible triangulation of the polygonal domain Ω

to show $\Sigma_h \subset H(\mathrm{div}, \Omega)$:

$$\mathrm{div}\, \tau_h|_T = 2c_T$$

$$\tau_h|_T \cdot n = \begin{pmatrix} a_T \\ b_T \end{pmatrix} \cdot n + c_T \underbrace{(x \cdot n)}_{\text{const on every edge}}$$

$$= \text{const}$$

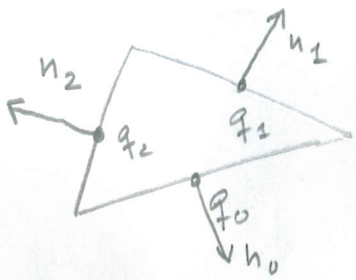


$x_1 \cdot n = x_2 \cdot n$

still to verify: trianglewise defined $\mathrm{div}\, \tau_h$ is the weak divergence of τ_h!

$$\int_\Omega \tau_h \cdot \nabla \varphi \, dx = \sum_{T \in \mathcal{J}_h} -\int_T \mathrm{div}\, \tau_h|_T \, \varphi \, dx + \sum_{E \in \mathcal{E}_h^\circ} \int_E \underbrace{[\tau_h \cdot n]_E}_{=0} \varphi \, da$$

- **degrees of freedom for the Raviart - Thomas element:**

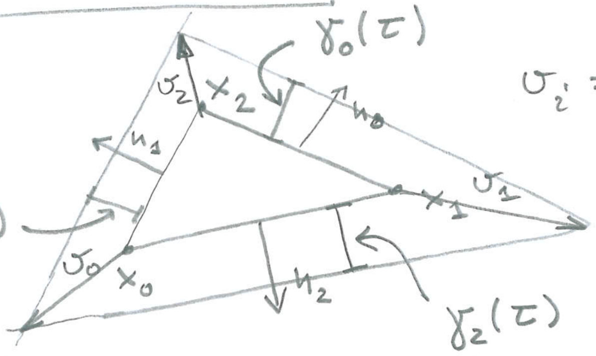


$$\boxed{\begin{matrix} \gamma_i(\tau) = \tau(q_i) \cdot n_i \\ i = 0, 1, 2 \end{matrix}}$$

$$\boxed{\begin{pmatrix} a_T \\ b_T \end{pmatrix} \cdot n_i + c_T \, q_i \cdot n_i = \gamma_i \qquad i = 0, 1, 2}$$

(linear system of eq.)

- **unique solvability:**



$v_i = \tau(x_i)$

$$v_1 \cdot n_0 = v_2 \cdot n_0$$
$$v_1 \cdot n_2 = v_0 \cdot n_2$$
$$v_0 \cdot n_1 = v_2 \cdot n_1$$

$\rightsquigarrow$ v_i at x_i can be constructed (see sketch) such that

$$v_i \cdot n_j = \gamma_j(\tau)$$

def $\boxed{v \in \mathcal{S}_1^2}$ piecewise affine field with $v(p_i) = v_i$

$\Rightarrow$ $v \cdot n_j$ is linear on the edge $\underset{\text{construction}}{\Rightarrow}$ $\boxed{v \cdot n_j \equiv \gamma_i}$

$$v(x) \overset{!}{=} \binom{a_T}{b_T} + c_T x = \underbrace{\binom{a_T}{b_T} - c_T x_0}_{\binom{a_T'}{b_T'} = v_0} + c_T (x - x_0)$$

choose, such that $v(x_1) = v_1$ (possible due to the constant normal component)

$\leadsto$ by construction: $v(x_2) = v_2$

($\checkmark$)

4.8. Lemma ($\operatorname{div} : \Sigma_e \to M_e$ surjective)

The mapping $\operatorname{div} : \Sigma_e \to M_e$ is surjective, i.e.
for each $p_e \in M_e$ there exists a $\tau_e \in \Sigma_e$ with $\operatorname{div} \tau_e|_T = p_e|_T$
and $\|\tau_e\|_{H(\operatorname{div},\Omega)} \le C_1 \|p_e\|_{0,2,\Omega}$

Pf $\boxed{1^{st}\text{ step}}$ choose convex domain $\tilde{\Omega} \supset \Omega$ and expand p_e onto $\tilde{\Omega}$ via $p_e|_{\tilde{\Omega}\setminus\Omega} \equiv 0$, then

$$\boxed{\begin{aligned} \Delta u &= p_e \quad \text{in } \tilde{\Omega} \\ u &= 0 \quad \text{on } \partial\tilde{\Omega} \end{aligned}}$$

has a unique solution $u \in H^2(\tilde{\Omega}) \cap H^1_0(\tilde{\Omega})$

due to the convexity

$\boxed{2^{nd}\text{ step}}$ $\tau := \nabla u \in H^1(\Omega)$ $\quad \operatorname{div} \tau = p_e$ on $\tilde{\Omega}$

define $\tau_e \in \Sigma_e$ via $\int_E \tau_e \cdot n \, da = \int_E \tau \cdot n \, da \quad \forall_{E \in \mathcal{E}_e}$

$$\Rightarrow \int_T p_e \, dx = \int_T \operatorname{div} \tau \, dx$$

$$= \int_{\partial T} \tau \cdot n \, da = \int_{\partial T} \tau_e \cdot n \, da = \int_T \underbrace{\operatorname{div} \tau_e}_{= \text{const on } T} dx$$

$$\Rightarrow \boxed{\operatorname{div} \tau_e = p_e \text{ on all } \overset{\circ}{T} \quad \forall_{T \in \mathcal{J}_e}}$$

$\boxed{3^{rd}\text{ step}}$ $\quad \hat\tau_e(\hat x) = \tau_e(x) \qquad x = F_T(\hat x)$ reference map $\hat T \to T$

$$\|\tau_e\|^2_{0,2,T} \le C h^2 \|\hat\tau_e\|_{0,2,\hat T}$$

$$\underset{\boxed{\text{equivalence of norms}}}{\le} C h^2 \sum_{E\in\mathcal{E}_e(T)} \Big(\underbrace{\int_{\hat E} \hat\tau_e\cdot\hat n\, d\hat a}\Big)^2$$

$$= \int_E \tau_e\cdot n\, da = \int_E \tau\cdot n\, da$$

$$= \int_{\hat E} \hat\tau\cdot\hat n\, d\hat a$$

$\hat n(\hat x) = n(x)$ ($\hat n$ no longer normal in general!)

$$\underset{\boxed{\text{trace thm}}}{\le} C h^2 \|\hat\tau\|^2_{1,2,\hat T} \underset{\text{scaling}}{\le} C \|\tau\|^2_{1,2,T}$$

$$\Rightarrow \quad \|\tau_e\|^2_{H(div,\Omega)} = \underbrace{\|\tau_e\|^2_{0,2,\Omega}}_{\text{see above}} + \|\underbrace{div\,\tau_e}_{=p_e}\|^2_{0,2,\Omega}$$

$$\le C\sum_T \|\tau\|^2_{1,2,T} \qquad = \|p_e\|^2_{0,2,\Omega}$$

$$\le C\|\tau\|^2_{1,2,\Omega} \le C\|u\|^2_{2,2,\tilde\Omega}$$

$$\le C\|p_e\|^2_{0,2,\tilde\Omega}$$

$p_e|_{\tilde\Omega\setminus\Omega} \equiv 0$

$$\Rightarrow \quad \|\tau_e\|_{H(div,\Omega)} \le C\|p_e\|_{0,2,\Omega} \qquad \square$$

This immediately implies the inf-sup-condition:

$$\sup_{\tau_e\in X_e} \frac{\int_\Omega (div\,\tau_e)p_e\, dx}{\|\tau_e\|_{H(div,\Omega)}} \underset{\boxed{\tau_e \text{ from above with } div\,\tau_e = p_e}}{\ge} \frac{1}{C}\frac{\int_\Omega p_e^2\, dx}{\|p_e\|_{0,2,\Omega}} = \frac{1}{C}\|p_e\|_{0,2,\Omega}$$

(✓)

This implies:

4.9. Thm (existence of discrete solutions)

For $f \in L^2(\Omega)$, there exists a unique discrete solution $(\sigma_h, u_h) \in \Sigma_h \times M_h$ of $(P_{S,h})$

4.10 Thm (error estimate)

If $u \in H^2(\Omega)$, $f \in H^1(\Omega)$ and $\mathcal{T}_h$ a regular admissible triangulation, then the following estimate holds

$$\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)} + \|u - u_h\|_{0,2,\Omega} \le C\, h \left(\|u\|_{2,2,\Omega} + \|f\|_{1,2,\Omega}\right)$$

Pf

$$\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)} + \|u - u_h\|_{0,2,\Omega} \overset{4.6.}{\le} C\left(\inf_{\tau_h \in \Sigma_h} \|\sigma - \tau_h\|_{H(\mathrm{div},\Omega)} + \inf_{p_h \in M_h} \|u - p_h\|_{0,2,\Omega}\right)$$

1st step

$$\|u - \mathcal{I}_0 u\|_{0,2,\Omega} \le C\, h\, \|u\|_{1,2,\Omega}$$

p.w. constant interpolation with $\mathcal{I}_0 u|_T = \fint_T u\,dx$

2nd step

$\mathcal{I}_{RT}$: Raviart Thomas interpolation with

$$\int_E (\mathcal{I}_{RT}\tau - \tau)\cdot n\,da = 0 \quad \forall E \in \mathcal{E}(T),\ T \in \mathcal{T}_h$$

$$\Rightarrow \int_T \mathrm{div}(\tau - \mathcal{I}_{RT}\tau)\, p_h\,dx = 0 \quad \forall p_h \in M_h$$

For $\tau|_T = \text{const} \Rightarrow \mathcal{I}_{RT}\tau = \tau$

$$\Rightarrow \|\tau - \mathcal{I}_{RT}\tau\|_{0,2,T} \overset{\uparrow}{\le} C\, h\, |\tau|_{1,2,T}$$

compare Lagrange-interpolation [Bramble-Hilbert-Lemma (see exercise)]

$$\Rightarrow \|\tau - \mathcal{I}_{RT}\tau\|_{0,2,\Omega} \le C\, h\, |\tau|_{1,2,\Omega}$$

$\boxed{3^{rd}\text{ step}}$ **estimate of the divergence:**

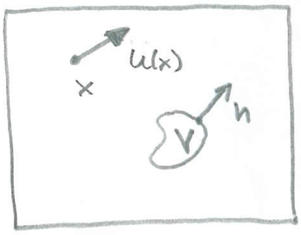
$$\int_\Omega \operatorname{div}(\tau - \mathcal{I}_{RT}\tau)\, \underset{\in M_h}{p_h}\, dx \overset{\text{see above}}{=} 0$$

$\Rightarrow$ $\operatorname{div} \mathcal{I}_{RT}\tau$ is the L^2-projection of $\operatorname{div}\tau$ on M_h

$$\Rightarrow \|\operatorname{div}(\tau - \mathcal{I}_{RT}\tau)\|_{0,2,\Omega} \le \inf_{p_h \in M_h} \|\underbrace{\operatorname{div}\tau}_{= f \in H^1(\Omega)} - p_h\|_{0,2,\Omega}$$

$$\underset{\text{see above}}{\le} C\, h\, \|f\|_{1,2,\Omega} \qquad \square$$

Next, we study a very basic model for a (highly) viscous stationary flow:



$u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ velocity field

particle paths: $(x(t))_{t\in\mathbb{R}}$ with $\dot{x}(t) = u(x(t))$

incompressibility: $\int_{\partial V} u \cdot n\, da = 0$ (in/out flow) for all testvolumes $V \subset \Omega$

$\overset{\text{Gauß thm}}{\Longrightarrow}$ $\int_V \operatorname{div} u\, dx = 0$

$\overset{\forall\text{ testvolumes }V}{\Longrightarrow}$ $\operatorname{div} u = 0$ in Ω

strong formulation:

acceleration force

$$(P_{Stokes}) \quad \boxed{\begin{aligned} -\Delta u + \nabla p &= f && \text{in } \Omega \\ \operatorname{div} u &= 0 && \text{in } \Omega \\ u &= u^\partial && \text{on } \partial\Omega \end{aligned}} \quad \text{(Stokes eqs)}$$

p determined only up to a constant, thus we require $\int_\Omega p\, dx =$

$$\int_{\partial\Omega} u^\partial \cdot n\, da = \int_{\partial\Omega} u\cdot n\, da = \int_\Omega \operatorname{div} u\, dx = 0$$

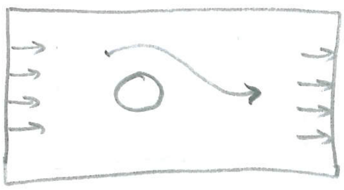
Hence we ask for a solution (u,p) of (P_{Stokes}) with

$u : \overline{\Omega} \to \mathbb{R}^d$, $p : \Omega \to \mathbb{R}$ with $\fint_\Omega p \, dx = 0$

and $u = u^\partial$ on $\partial\Omega$ with $\int_{\partial\Omega} u^\partial \cdot n \, da = 0$.

$(u,p) \in C^2(\Omega) \cap C^0(\overline{\Omega}) \times C^1(\Omega)$ fulfilling these conditions and solving (P_{Stokes}) is denoted a classical solution.

physical modelling (very short):



X Lagrangian coordinate (fixed, reference)

x Eulerian coordinate (observer coordinates)

$x(t) = \Phi(t, X)$ flow fct

$\frac{d}{dt} \varphi(t, x(t)) = \partial_t \varphi + \dot{x} \cdot \nabla_x \varphi =: \frac{D}{\partial t} \varphi$ material derivative along a particle path

motion: $\dot{x}(t) = u(t, x(t))$ velocity

$\varrho \equiv$ const density

conservation laws: • mass conservation $\overset{\varrho \equiv \text{const}}{\longleftrightarrow}$ incompressibility

$\longleftrightarrow$ $\operatorname{div} u = 0$

• conservation of momentum

$$\int_V \varrho \underbrace{\frac{D}{\partial t} v}_{\substack{\text{acceleration along} \\ \text{the particle path} \\ (\dot{x}(t) = v(t, x(t)))}} dx = \int_{\partial V} \underbrace{t(x,n) \, da}_{\substack{\text{forcing on the} \\ \text{boundary } \partial V \\ \text{with normal } n}} + \int_V \varrho f \, dx$$

f: external acceleration

key inside: $t(x,n) = T(x)\, n$

T: stress tensor $(\mathbb{R}^{d,d})$

material law: Newtonian fluid model

$$T = -\underset{\substack{\uparrow\\ \text{physical pressure}}}{q}\,\mathbb{1} + \underbrace{2\mu D}_{\text{friction component}} \qquad D = \tfrac{1}{2}(\nabla u + \nabla u^T)$$

$$\left(D_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)\right)_{i,j=1,\dots,d}$$

Gauß thm
$\Rightarrow$ $$\int \rho \frac{D}{\partial t} u - \operatorname{div} T - \rho f \, dx = 0$$

$\forall$ testvolumes
$\Rightarrow$ pointwise: $$\rho(\partial_t u + \underbrace{u\cdot\nabla u}_{\left(\sum_{i=1}^d u_i \partial_i u_j\right)_{j=1,\dots,d}}) - 2\mu \operatorname{div} D + \operatorname{div}(q\mathbb{1}) = \rho f$$

$$\operatorname{div} D = \frac{1}{2}\left(\sum_{i=1}^d \partial_{x_i} D_{ji}\right)_{j=1,\dots,d} = \frac{1}{2}\left(\sum_{i=1}^d \partial_{x_i}\partial_{x_i} u_j + \partial_{x_i}\partial_{x_j} u_i\right)_{j=1,\dots,d}$$

$$= \frac{1}{2}\Delta u + \frac{1}{2}\nabla \underbrace{\operatorname{div} u}_{=0} = \frac{1}{2}\Delta u$$

$$\operatorname{div}(q\mathbb{1}) = \left(\sum_{i=1}^d \partial_{x_i} q\, \delta_{ij}\right)_{j=1,\dots,d} = \nabla q$$

$$\Rightarrow \boxed{\begin{pmatrix} \partial_t u + u\cdot\nabla u - \nu\Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{pmatrix}}$$

with $\nu = \frac{\mu}{\rho}$, $p = \frac{q}{\rho}$

incompressible, time dependent Navier-Stokes eqs

$$\left.\begin{array}{l} \nu \gg 1 \quad \text{(high viscosity)} \\ \partial_t u = 0 \end{array}\right\} \xrightarrow[\text{rescaling}]{\text{neglect the nonlinearity}} \boxed{\begin{array}{l} -\Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{array}}$$

Stokes eqs

weak formulation:

$$a(u,v) := \int_\Omega \sum_{i=1}^{d} \nabla u_i \cdot \nabla v_i \, dx \qquad \forall u,v \in H^{1,2}(\Omega)^d$$

$$b(v,q) := -\int_\Omega (\operatorname{div} v)\, q \, dx \qquad \forall v \in H^{1,2}(\Omega),\ q \in L^2(\Omega)$$

$\boxed{\text{for simplicity } u^\partial \equiv 0}$

then P_{Stokes} implies:

$$(\tilde{P}_{Stokes}) \qquad a(u,v) + b(v,p) = \langle f, v \rangle \qquad f \in H^{-1,2}(\Omega)^d$$

$$b(u,q) = 0$$

$$\forall v \in H_0^{1,2}(\Omega)^d =: X$$

where $L_0^2 = \{ q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0 \}$ $\qquad q \in L_0^2 =: M$

elsewise, for $u^\partial \in H_0^{1,2}(\Omega)^d$: $\qquad \langle \tilde{f}, v \rangle = \langle f, v \rangle - a(u^\partial, v)$

$$\underset{H_0^{1,2}(\Omega)^d}{u = \underset{\cap}{u^0} + u^\partial} \qquad \langle g, q \rangle = -b(u^\partial, q)$$

and $\qquad a(u^0, v) + b(v,p) = \langle \tilde{f}, v \rangle \qquad \forall v \in X$

$$b(u,q) = \langle g, q \rangle \qquad q \in M$$

4.11 Thm (Existence of weak solutions)

Ω Lipschitz domain, bounded, $f \in L^2(\Omega)$,
then there exists a unique weak solution
(u,p) with $u = u^0 + u^\partial$ ($u^0 \in H_0^{1,2}(\Omega)$, $u^\partial \in H^{1,2}(\Omega)$)
and $p \in L_0^2(\Omega) = \{ q \in L^2(\Omega) \mid \int_\Omega q \, dx = 0 \}$ of $(\tilde{P}_{Stokes})$.

Pf

bounded

$$a(u,v) \le C \underbrace{\|u\|_{1,2,\Omega}}_{=\left(\sum_{i=1}^{d} \|u_i\|^2_{1,2,\Omega}\right)^{\frac{1}{2}}} \|v\|_{1,2,\Omega}$$

$$b(v,q) \le C \|\operatorname{div} v\|_{0,2,\Omega} \|q\|_{0,2,\Omega} \le C \|v\|_{1,2,\Omega} \|q\|_{0,2,\Omega}$$

$$a(u,u) \ge C |u|^2_{1,2,\Omega} \overset{\text{Poincaré}}{\ge} c \|u\|^2_{1,2,\Omega} \quad \text{(coercive)}$$

inf-sup-condition:

- without pf: $\left[\forall q \in L^2_0(\Omega) \; \exists v \in H^{1,2}_0(\Omega) \; \operatorname{div} v = q \text{ a.e. and } \|v\|_{1,2,\Omega} \le C \|q\|_{0,2,\Omega} \right]$

$$\Rightarrow \sup_{\tilde{v} \in X} \frac{b(\tilde{v},q)}{\|\tilde{v}\|_X} \ge \frac{b(v,q)}{\|v\|_X} = \frac{\int q^2 \, dx}{\|v\|_X}$$

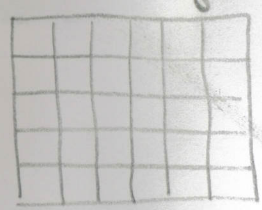
$$\ge \frac{\|q\|^2_{0,2,\Omega}}{C \|q\|_{0,2,\Omega}} = \frac{1}{C} \|q\|^2_{0,2,\Omega}$$

- claims follows with Thm 4.5 $\square$

numerical approximation:

$\boxed{1^{\text{st}} \text{ attempt}}$ (i) rectangular elements K

$h = \frac{1}{N}$, $d = 2$



$$X_h = \left\{ v_h \in C^0(\bar{\Omega})^2 \;\middle|\; v_h|_K \text{ bilinear}, \; v_h|_{\partial\Omega} = 0 \right\}$$

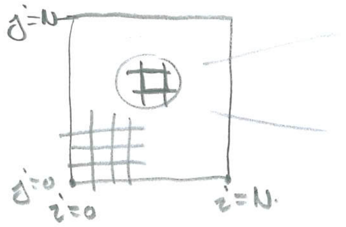
$$M_h = \left\{ q_h \in L^2_0(\Omega) \;\middle|\; q_h|_K = \text{const} \right\}$$

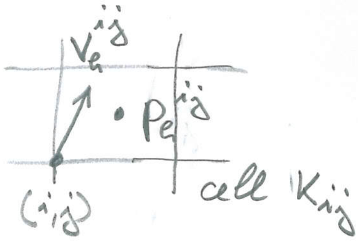
pb there exists $p_h \in M_h$, $p_h \neq 0$ with

$$b(v_h, p_h) = -\int_\Omega (\operatorname{div} v_h)\, p_h \, dx = 0 \quad \forall v_h \in X_h$$

$\Rightarrow$ discrete inf-sup condition violated!

to this end:





$$p_h|_{K_{ij}} = p_h^{ij}$$

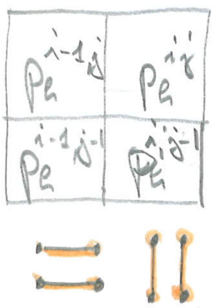
$$b(v_h, p_h) = -\sum_{i,j=0,\dots,N-1} \int_{K_{ij}} (\operatorname{div} v_h)\, p_h \, dx$$

$$= -\sum_{i,j=0,\dots,N-1} p_h^{ij} \int_{\partial K_{ij}} v_h \cdot n \, da$$

$$= -\frac{1}{2} h^2 \sum_{i,j=0,\dots,N-1} p_h^{ij} \frac{1}{h} \Big(v_{h1}^{i+1,j} + v_{h1}^{i+1,j+1} - v_{h1}^{ij} - v_{h1}^{i,j+1} + v_{h2}^{i+1,j+1} + v_{h2}^{i,j+1} - v_{h2}^{ij} - v_{h2}^{i+1,j} \Big)$$

summation by parts

$$= h^2 \sum_{i,j=1}^{N-1} v_h^{ij} \cdot \underbrace{\nabla^h p_h^{ij}}$$



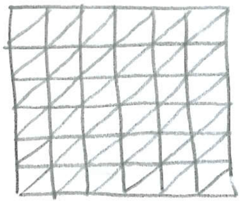
$$= \frac{1}{2h} \begin{pmatrix} (p_h^{ij} + p_h^{i,j-1}) - (p_h^{i-1,j} + p_h^{i-1,j-1}) \\ (p_h^{ij} + p_h^{i-1,j}) - (p_h^{i,j-1} + p_h^{i-1,j-1}) \end{pmatrix}$$

$\Rightarrow$ $b(v_h, p_h) = 0 \;\; \forall v_h \in X_h \iff \nabla^h p_h^{ij} = 0 \;\; \forall_{i,j=1,\dots,N-1}$

choose $p_h^{ij} = (-1)^{ij}$ (spurious pressure mode),

then $\nabla^h p_h^{ij} = \frac{1}{2h}\begin{pmatrix} 0-0 \\ 0-0 \end{pmatrix} = 0 \qquad \|p_h\|_{0,\Omega} = 1$

(ii) $\mathcal{T}_h$



$h = \frac{1}{N}$

$\Omega = [0,1]^2$

$N \bmod 3 = 0$

$$X_h = \{ v_h \in C^0(\bar{\Omega})^2 \mid v_h|_T \in \mathcal{P}_1^2 ,\ v_h|_{\partial\Omega} = 0 \}$$

$$M_h = \{ q_h \in C^0(\bar{\Omega}) \mid p_h|_T \in \mathcal{P}_1 ,\ p_h \in L_0^2(\Omega) \}$$

$$b(v_h, p_h) = - \int_\Omega \operatorname{div} v_h \, p_h \, dx = - \sum_{T \in \mathcal{T}_h} \underbrace{(\operatorname{div} v_h)|_T}_{= \text{const}} \int_T p_h \, dx$$

$$= - \sum_{T \in \mathcal{T}_h} (\operatorname{div} v_h)|_T \frac{|T|}{3} \left(p_h(x_0^T) + p_h(x_1^T) + p_h(x_2^T) \right)$$

$(x_i^T)_{i=0,..2}$ vertices of T

Now choose: $\quad p_h((ih, jh)) = \begin{cases} 0 & ; (i+j) \bmod 3 = 0 \\ 1 & ; (i+j) \bmod 3 = 1 \\ -1 & ; \quad \text{—"—} \quad = 2 \end{cases}$

$$\begin{array}{cccccc} 0 & 1 & -1 & 0 & 1 & -1 \\ -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 1 & -1 \end{array}$$

$\Rightarrow \sum_{i=0,1,2} p_h(x_i^T) = 0$ (spurious pressure mode)

$\Rightarrow \quad b(v_h, p_h) = 0 \quad \forall v_h \in X_h$

Now we consider pairings (X_h, M_h) which fulfill the discrete inf-sup condition:

the pb with above pairs (X_h, M_h) is, that X_h is not rich enough to control the spurious pressure mode p_h!

$\rightsquigarrow$ $\boxed{\text{suitable enrichment of the velocity space required!}}$

on a simplicial mesh: $\boxed{d=2}$

$$b_T = \prod_{i=0}^{2} \lambda_i \quad (\text{bubble})$$

$$X_h = \{ v_h \in C^0(\bar\Omega)^d \mid v_h|_{\partial\Omega} = 0 ,\ v_h|_T \in (\mathcal{P}_1 \oplus \operatorname{span} b_T)^2 \}$$

$$M_h = \{ q_h \in C^0(\bar\Omega)^d \mid q_h|_T \in \mathcal{P}_1 ,\ \int_\Omega q_h \, dx = 0 \}$$

(minielement)

4.12. Lemma $\boxed{\text{The minielement fulfills the discrete inf sup condition.}}$

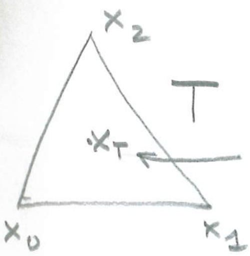
Pf recall from Thm 2.9: $\exists\ \mathcal{I}_h^0 : H_0^{1,2}(\Omega)^d \to \mathcal{V}_h^d$

with $|u - \mathcal{I}_h^0 u|_{m,p,\Omega} \le C h^{\ell - m} |u|_{\ell,p,\Omega}$

where $\mathcal{V}_{h,0}$ p.w. affine, continuous FE space with 0 bd.cond.

$m < \ell \le 2$, $2 - \frac{d}{2} > 0$ for $d=2$, and $|\mathcal{I}_h^0 u|_{m,2,\Omega} \le |u|_{m,2,\Omega}$

now consider a single triangle and define:



centre of mass

$$v_h(x_i) = (\mathcal{I}_h^0 v)(x_i) \quad \text{AND}$$

$$\circledast \quad \fint_T v_h \, dx = \fint_T v \, dx$$

possible due to bubble

in particular:

$$v_h|_T(x) = \sum_{j=0}^{2} v_h(x_j) \lambda_j(x) + 27 \lambda_0(x)\lambda_1(x)\lambda_2(x) \Big(v_h(x_T) - \frac{1}{3} \sum_{i=0}^{2} v_h(x_i) \Big)$$

barycentric coordinates $\lambda(x)$

$\Rightarrow$ adjust $v_h(x_T)$ corresponding to fulfill $\circledast$:

$$v_h(x_T) = \Big(\fint_T 27 \lambda_0 \lambda_1 \lambda_2 \, dx \Big)^{-1} \Big(\fint_T v - \mathcal{I}_h^0 v \, dx \Big) + \fint_T \mathcal{I}_h^0 v \, dx$$

now define $\Pi_h : X = H_0^{1,2}(\Omega)^d \to X_h$

$$\Pi_h v = v_h$$

To show: (i) $\int_\Omega \operatorname{div} \Pi_h v \, q_h \, dx = \int_\Omega \operatorname{div} v \, q_h \, dx$

(ii) $\| \Pi_h v \|_{1,2,\Omega} \leq C \| v \|_{1,2,\Omega}$

$\overset{4.7.}{\Longrightarrow}$ claim (Fortin)

ad (i):

$$\int_\Omega \operatorname{div} v_h \, q_h \, dx = \sum_T \int_T \operatorname{div} v_h \, q_h \, dx$$

$$= \sum_T \Big(\underbrace{\int_{\partial T} v_h \cdot n \, q_h \, da}_{=0} - \int_T v_h \underbrace{\operatorname{grad} q_h}_{= \text{const}} dx \Big)$$

$$= -\sum_T \int_T v \cdot \operatorname{grad} q_h \, dx = \sum_T \Big(\underbrace{-\int_{\partial T} v \cdot n \, q_h \, da}_{=0} + \int_T \operatorname{div} v \, q_h \, dx \Big)$$

$$= \int_\Omega \operatorname{div} v \, q_h \, dx$$

ad (ii):

$$|v_h|_{1,2,T}^2 \leq \Big| \mathcal{I}_h v + 27 \lambda_0 \lambda_1 \lambda_2 \underbrace{\Big(v_h(x_T) - \fint_T \mathcal{I}_h v \Big)}_{\left(\fint_T 27\lambda_0\lambda_1\lambda_2 \, dx\right)^{-1} \left(\fint_T v - \mathcal{I}_h v \, d\ldots\right)} \Big|_{1,2,T}^2$$

$$\leq C \Big(|\mathcal{I}_h v|_{1,2,T}^2 + |v - \mathcal{I}_h v|_{1,2,T}^2 + \frac{1}{h^2} \| v - \mathcal{I}_h v \|_{0,2,T}^2 \Big) \leq C \Big(\| v \|_{1,2,\omega_T}^2 + \frac{1}{h^2} h^2 \| v \|_{1,2,\omega_T}^2 \Big)$$

differentiation of bubble terms

$$\Rightarrow \quad |v_h|_{1,2,\Omega} \leq C \| v \|_{1,2,\Omega}$$

$\square$

- $\boxed{\inf_{p_h \in M_h} \| p_h - p \| \ ?}$ $\quad M_h = \{ q_h \in L^2_0(\Omega) \mid q_h|_T \in P_1 \}$

if $p \in L^2_0$, then $p_h = \mathcal{I}_h p$ not necessarily ← local projection on the p.w. affine, cont. FE space

fulfill $\int_\Omega p_h \, dx = 0$

but: $\int_\Omega p_h \, dx = \int_\Omega (p - p_h) \, dx \le |\Omega|^{\frac{1}{2}} \underbrace{\| p - p_h \|_{0,2,\Omega}}_{\le C h \| p \|_{1,2,\Omega}}$

- define $\tilde{p}_h = \mathcal{I}_h p - \fint_\Omega \mathcal{I}_h p \, dx$

$\rightarrow \| \tilde{p}_h - p \|_{0,2,\Omega} \le \| \mathcal{I}_h p - p \|_{0,2,\Omega} + \| \fint_\Omega \mathcal{I}_h p - p \, dx \|_{0,2,\Omega}$

$\le C h \| p \|_{1,2,\Omega}$

-

4.13 **Thm** (error estimates for the mini element)

Assumptions as above, (u,p) solves $(\tilde{P}_{Stokes})$, $u \in H^{2,2}(\Omega)$, $p \in H^{1,2}_{loc}$

$$\|u - u_h\|_{1,2,\Omega} + \|p - p_h\| \leq C h \left(\|u\|_{2,2,\Omega} + \|p\|_{1,2,\Omega}\right)$$

where (u_h, p_h) solves

$$(\tilde{P}_{Stokes,h}) \qquad \begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= \langle f, v_h \rangle \quad \forall v_h \in X_h \\ b(u_h, q_h) &= 0 \quad \forall q_h \in M_h \end{aligned}$$

$$u_h = u_h^0 + u_h^\partial \quad (u_h^\partial = u^\partial), \; u_h^0 \in X_h .$$

- another admissible element:

$\boxed{\text{Taylor-Hood-element}}$

$$X_h = \{ v_h \in C^0(\bar{\Omega}) \mid v_h|_{\partial\Omega} = 0, \; v_h|_T \in P_2^d \; \forall T \in J_h \}$$

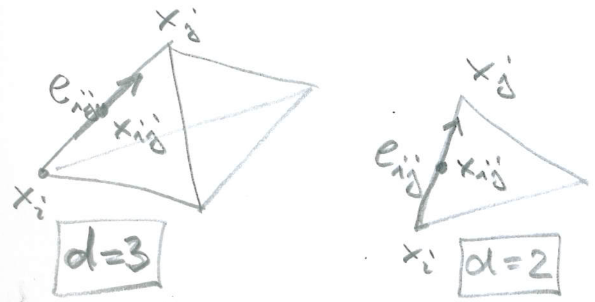
$$M_h = \{ p_h \in L^2_0(\bar{\Omega}) \cap C^0(\bar{\Omega}) \mid p_h|_T \in P_1 \; \forall T \in J_h \}$$

4.14 **Lemma** (discrete inf-sup condition)

- The Taylor-Hood-element for $d = 2, 3$ fulfills the discrete inf-sup-condition, if every simplex has at least d lin. independent edges in Ω.

Pf (i) we show:

$$\sup_{v_h \in X_h} \frac{\int_\Omega q_h \operatorname{div} v_h \, dx}{\|v_h\|_{1,2,\Omega}} \geq c \left(\sum_{T \in J_h} h_T^2 \|q_h\|_{1,2,T}^2 \right)^{\frac{1}{2}}$$



$$e_{ij} = \frac{x_i - x_j}{|x_i - x_j|} \qquad \text{(direction arbitrary)}$$

$$l_{ij} = |x_i - x_j|$$

now define v_e: $\quad v_e(x_i) = 0$

$v_e|_{\partial\Omega} = 0$, $\boxed{e_{ij} \text{ interior edge:}}$ $v_e(x_{ij}) = -\ell_{ij}^2 \, e_{ij} \, \partial_{e_{ij}} q_e$ $\Big\}$ uniquely defines in X_e

consider Lagrangian quadrature on T:

$$\int_T \underbrace{\phi(x)}_{\in \mathcal{P}_2} dx = \Big(\sum_{i<j} \omega_{ij} \phi(x_{ij}) + \sum_{i} \omega_i \phi(x_i) \Big) |T|$$

$\nabla q_e|_T \equiv \text{const}$

$$\Rightarrow \int_\Omega q_e \operatorname{div} v_e \, dx = - \sum_T \int_T v_e \cdot \nabla q_e \, dx$$

$$= - \sum_T |T| \sum_{i<j} v_e(x_{ij}) \nabla q_e(x_{ij}) \, \omega_{ij}$$

$$= + \sum_T |T| \sum_{i<j} |\partial_{e_{ij}} q_e|^2 \ell_{ij}^2 \geq C \sum_T h_T^2 |q_e|^2_{1,2,T}$$

$\boxed{T \text{ has } d \text{ linear independent edges in } \Omega}$

Furthermore:

$$\| v_e \|^2_{0,2,T} \leq C_1 \max_{i<j} |v_e(x_{ij})|^2 \ell_{ij}^4 h_T^d \leq C_1 |\nabla q_e|_T|^2 h_T^{d+4}$$

$$\leq C_1 \, |q_e|^2_{1,2,T} \, h_T^4$$

$$\| \nabla v_e \|^2_{0,2,T} \leq C_1 \frac{1}{h_T^2} \max_{i<j} |v_e(x_{ij})|^2 \ell_{ij}^4 h_T^d$$

$$\underset{\text{see above}}{\leq} C_1 \, h_T^2 \, |q_e|^2_{1,2,T}$$

$$\Rightarrow \| v_e \|_{1,2,\Omega} \leq C_1 \Big(\sum_{T \in \mathcal{J}_e} h_T^2 |q_e|^2_{1,2,T} \Big)^{\frac{1}{2}}$$

and altogether:

$$\sup_{v_e \in X_e} \frac{\int_\Omega q_e \operatorname{div} v_e \, dx}{\| v_e \|_{1,2,\Omega}} \geq C_1 \Big(\sum_T h_T^2 |q_e|^2_{1,2,T} \Big)^{\frac{1}{2}}$$

(ii) Let $v \in H_0^{1,2}(\Omega)$ be a solution of $\operatorname{div} v = q_e$ with

$$\|v\|_{1,2,\Omega} \le C_1 \|q_e\|_{0,2,\Omega} \quad \Rightarrow$$

local projection

$$\sup_{v_e \in X_e} \frac{\int_\Omega q_e \operatorname{div} v_e}{\|v_e\|_{1,2,\Omega}} \ge \frac{\int_\Omega q_e \operatorname{div}(\mathcal{I}_e^0(v))\,dx}{\|\mathcal{I}_e^0 v\|_{1,2,\Omega}} \ge c\, \frac{\int_\Omega q_e \operatorname{div} \mathcal{I}_e^0(v)}{\|v\|_{1,2,\Omega}}$$

$$\boxed{\|\mathcal{I}_e^0 v\|_{1,2,\Omega} \le C_1 \|v\|_{1,2,\Omega}}$$

$$= c\, \underbrace{\frac{\int_\Omega q_e \operatorname{div} v}{\|v\|_{1,2,\Omega}}}_{\ge \beta \|q_e\|_{0,2,\Omega}} + c\, \underbrace{\frac{\int_\Omega q_e \operatorname{div}(\mathcal{I}_e^0 v - v)\,dx}{\|v\|_{1,2,\Omega}}}_{\cdots}$$

$$\ge \beta \|q_e\|_{0,2,\Omega} + c\, \underbrace{\frac{\int_\Omega (\mathcal{I}_e^0 v - v)\nabla q_e \,dx}{\|v\|_{1,2,\Omega}}}$$

$$\le \frac{\sum_T |q_e|_{1,2,T}\, h_T\, |v|_{1,2,\omega_T}}{\|v\|_{1,2,\Omega}}$$

$$\le \Big(\sum_T |q_e|_{1,2,T}^2\, h_T^2\Big)^{\frac{1}{2}}$$

$$\ge \beta \|q_e\|_{0,2,\Omega} - c \Big(\sum_T |q_e|_{1,2,T}^2\, h_T^2\Big)^{\frac{1}{2}}$$

$$\ge \beta \|q_e\|_{0,2,\Omega} - \frac{c}{C_1} \sup_{v_e \in X_e} \frac{\int_\Omega q_e \operatorname{div} v_e \,dx}{\|v_e\|_{1,2,\Omega}}$$

$$\Rightarrow \quad \sup_{v_e \in X_e} \frac{\int_\Omega q_e \operatorname{div} v_e \,dx}{\|v_e\|_{1,2,\Omega}} \ge \beta \Big(1 + \frac{c}{C_1}\Big)^{-1} \|q_e\|_{0,2,\Omega}$$

$\square$

4.15 Thm

$$\boxed{\begin{array}{l}\text{Under the assumptions of 4.14, and}\\ \text{for } (u,p) \text{ solution of } \tilde{P}_{\text{stokes}},\ u \in H^{3,2}(\Omega),\ p \in H^{2,2}(\Omega),\\ \text{and } (u_h, p_h) \text{ the Taylor-Hood-solution, we obtain}\\ \|u-u_h\|_{1,2} + \|p-p_h\|_{0,2} \leq C h^2 \left(\|u\|_{3,2,\Omega} + \|p\|_{2,2}\right)\end{array}}$$

Pf: follows from 4.6, 4.14 and remark on the pf of Thm 4.13. □

now we care about the linear algebra:

Find $u_h \in X_h$, $p_h \in M_h$ such that

$$(\tilde{P}_{s,h}) \quad \begin{array}{ll} a(u_h, v_h) + b(v_h, p_h) = \langle f, v_h \rangle & \forall v_h \in X_h \\ b(u_h, q_h) \qquad\qquad\quad\ = \langle g, q_h \rangle & \forall q_h \in M_h \end{array}$$

Basis representation: $u_h = \sum_{j=1,\dots N_u} \mathcal{U}_h^j v_h^j$ $\quad \{v_h^1, \dots v_h^{N_u}\}$ basis of X_h

$\mathcal{U}_h = (\mathcal{U}_h^j)_{j=1,\dots N_u}$ $\qquad p_h = \sum_{j=1,\dots N_p} P_h^j q_h^j$ $\quad \{q_h^1, \dots q_h^{N_p}\}$ basis of M_h

$P_h = (P_h^j)_{j=1,\dots N_p}$

$$A_h = \left(a(v_h^j, v_h^i)\right)_{\substack{i=1,\dots N_u\\ j=1,\dots N_u}} \qquad A_h \mathcal{U}_h \cdot \mathcal{W}_h = a(u_h, w_h)$$

$$B_h = \left(b(v_h^j, q_h^i)\right)_{\substack{i=1,\dots N_p\\ j=1,\dots N_u}} \qquad B_h V_h \cdot Q_h = b(v_h, q_h)$$

$$(\tilde{P}_{s,h}) \Leftrightarrow \begin{pmatrix} \boxed{A_h} & \boxed{B_h^T} \\ \boxed{B_h} & \boxed{0} \end{pmatrix} \begin{pmatrix} \mathcal{U}_h \\ P_h \end{pmatrix} = \begin{pmatrix} F_h \\ G_h \end{pmatrix} \qquad \begin{array}{l} F_h^i = \langle f, v_h^i \rangle \\ G_h^i = \langle g, q_h^i \rangle \end{array}$$

$\boxed{(N_u + N_p) \times (N_u + N_p) \text{ linear system of eqs}}$

$\boxed{\text{Thereby, we identify matrices and discrete operators!}}$

<u>associated norms:</u> $\underset{\mathbb{R}^{N_u}}{\|\mathcal{U}_h\|_{X_h}} = \|u_h\|_X$, $\underset{\mathbb{R}^{N_p}}{\|P_h\|_M} = \|p_h\|_M$

<u>mass matrix on M_h:</u> $M_h^p P_h \cdot Q_h = (p_h, q_h)_{M_h}$

$$\lambda_{max}(M_h^p) \leq C h_{max}^d \quad \lambda_{min}(M_h^p) \geq c\, h_{min}^d \quad \Rightarrow$$

$$c\, h_{min}^d \underset{\mathbb{R}^{N_p}}{\|P_h\|} \leq \|P_h\|_M \leq C h_{max}^d \|P_h\|$$

$$K(M_h^p) = \frac{\lambda_{max}(M_h^p)}{\lambda_{min}(M_h^p)} \quad \text{condition number} \quad \Rightarrow \quad \boxed{K(M_h^p) \leq C\left(\frac{h_{max}}{h_{min}}\right)^d}$$

$$A_h \mathcal{U}_h \cdot \mathcal{U}_h \geq \alpha \|\mathcal{U}_h\|_X$$

$$\boxed{\|\mathcal{U}_h\|_* := \sup \frac{\mathcal{U}_h \cdot \mathcal{V}_h}{\|\mathcal{V}_h\|_X}} \quad \Rightarrow \quad \boxed{\|A_h \mathcal{U}_h\|_* \geq \alpha \|\mathcal{U}_h\|_X} \quad \Rightarrow A_h \text{ pos. def.}$$

Furthermore $\boxed{\|A_h \mathcal{U}_h\|_* \leq C \|\mathcal{U}_h\|_X}$

bdness of $a(\cdot,\cdot)$

discrete inf-sup-condition $\Rightarrow$

$$\inf_{P_h \in M_h} \sup_{\mathcal{U}_h \in X_h} \frac{B_h^T P_h \cdot \mathcal{U}_h}{\|P_h\|_M \|\mathcal{U}_h\|_X} \geq \beta_h > 0$$

$$\Leftrightarrow \quad \boxed{\|B_h^T P_h\|_* \geq \beta_h \|P_h\|_M}$$

Furthermore $\boxed{\|B_h^T P_h\|_* \leq C \|P_h\|_M}$

bdness of $b(\cdot,\cdot)$

<u>Pb with $(\tilde{P}_{S,h})$:</u> $L_h = \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}$ is symmetric but <u>not</u> positive definit!

indeed $L_h \begin{pmatrix} 0 \\ P_h \end{pmatrix} \cdot \begin{pmatrix} 0 \\ P_h \end{pmatrix} = 0$

1st approach Penalty method

$$\underbrace{\begin{pmatrix} A_h & B_h^T \\ -B_h & \epsilon M_h^p \end{pmatrix}}_{L_h^\epsilon} \begin{pmatrix} U_h^\epsilon \\ P_h^\epsilon \end{pmatrix} = \begin{pmatrix} F_h \\ -G_h \end{pmatrix}$$

$$\Rightarrow \quad \epsilon M_h^p P_h^\epsilon = B_h U_h^\epsilon - G_h \quad \Rightarrow P_h^\epsilon = \frac{1}{\epsilon}(M_h^p)^{-1}(B_h U_h^\epsilon - G_h)$$

$$\Rightarrow \quad \boxed{\underbrace{\left(A_h + \frac{1}{\epsilon} B_h^T (M_h^p)^{-1} B_h\right)}_{\text{pos. def, symmetric}} U_h^\epsilon = F_h + \frac{1}{\epsilon} B_h^T (M_h^p)^{-1} G_h}$$

remark: can be solved by CG

4.16 **Proposition** assumptions as above, $\beta_h \geq c$, then

$$\| U_h - U_h^\epsilon \|_X + \| P_h - P_h^\epsilon \|_H \leq C \epsilon \, \| P_h \|_H$$

Pf

$$\boxed{\begin{aligned} A_h (U_h - U_h^\epsilon) + B_h^T (P_h - P_h^\epsilon) &= 0 \\ -B_h (U_h - U_h^\epsilon) - \epsilon M_h^p P_h^\epsilon &= 0 \end{aligned}} \quad (*)$$

$\Rightarrow$

$$\| P_h - P_h^\epsilon \|_H \leq \frac{1}{\beta_h} \| B_h^T (P_h - P_h^\epsilon) \|_* = \frac{1}{\beta_h} \| A_h (U_h - U_h^\epsilon) \|_*$$

$$\leq \frac{C}{\beta_h} \| U_h - U_h^\epsilon \|_X \quad (**)$$

$(*)_1 \cdot (U_h - U_h^\epsilon) \Rightarrow$

$$\alpha \| U_h - U_h^\epsilon \|_X^2 \leq A_h (U_h - U_h^\epsilon) \cdot (U_h - U_h^\epsilon) = -B_h^T (P_h - P_h^\epsilon) \cdot (U_h - U_h^\epsilon)$$

$$= -(P_h - P_h^\epsilon) \cdot B_h (U_h - U_h^\epsilon)$$

$$= \quad \epsilon M_h^p P_h^\epsilon \cdot (P_h - P_h^\epsilon)$$

$$= \underbrace{-\epsilon\, M_h^p (P_h - P_h^\epsilon)\cdot(P_h - P_h^\epsilon)}_{\leq 0} + \epsilon\, M_h^p P_h \cdot (P_h - P_h^\epsilon)$$

$$\leq \epsilon\, \|P_h - P_h^\epsilon\|_M \|P_h\|_M$$

$$\overset{(**)}{\leq} \frac{\epsilon C}{c} \|U_h - U_h^\epsilon\| \|P_h\|_M \quad \Longrightarrow \quad \|U_h - U_h^\epsilon\|_X \leq C' \epsilon \|P_h\|_M$$

$$\underset{(**)}{\Longrightarrow} \|P_h - P_h^\epsilon\|_M \leq C\epsilon \|P_h\|_M \qquad \square$$

2^{nd} approach

elimination of U_h from $(\tilde{P}_{S,h})$:

$$\oplus \quad \boxed{A_h U_h = F_h - B_h^T P_h} \Rightarrow$$

$$U_h = A_h^{-1}(F_h - B_h^T P_h) \xrightarrow{-B_h U_h = -G_h}$$

$$\boxed{B_h A_h^{-1} B_h^T P_h = B_h A_h^{-1} F_h - G_h} \quad (***)$$

$S_h := B_h A_h^{-1} B_h^T$ is called Schur complement or Uzawa matrix

$$\left.\begin{array}{l} A_h \text{ pos. def, bd} \\ B_h^T \text{ bd, injective} \end{array}\right\} \Rightarrow S_h \text{ pos. def., bd}$$

$$\uparrow$$

$$\|B_h^T P_h\|_* \geq \beta_h \|P_h\|_M$$

Algorithm:
- Solve $(***)$ via iterative methods:
  - with on the fly multiplication wie B_h^T & B_h
  - in each step $A_h^{-1} W_h$ requires the solution of a system $A_h V_h = W_h$ again via and iterative (or direct) solver
- Finally solve $\oplus$ for U_h

4.17 **Proposition** (condition number of S_e)

$$\kappa(S_e) \leq \frac{C^2}{\alpha\beta_h} \kappa(M_e^p) \leq C\left(\frac{h_{max}}{h_{min}}\right)^d \quad (\text{if } \beta_h \geq \beta > 0)$$

Pf (i) Q_e eigenvect corresponding to $\lambda_{min}(S_e)$ $\Rightarrow$

$$\lambda_{min}(S_e)\|Q_e\|^2 = S_e Q_e \cdot Q_e = \underbrace{A_e^{-1}B_e^T Q_e}_{=:W_e} \cdot B_e^T Q_e$$

$$= W_e \cdot A_e W_e \geq \alpha \|W_e\|_X^2$$

$$\geq \frac{\alpha}{C^2}\|A_e W_e\|_*^2 = \frac{\alpha}{C^2}\|B_e^T Q_e\|_*^2 \geq \frac{\alpha\beta}{C^2}\|Q_e\|_M^2$$

$\boxed{\|A_e W_e\|_* \leq C\|W_e\|_X}$ $\quad$ $\boxed{\|B_e^T Q_e\|_* \geq \beta\|Q_e\|_M}$

$$\geq \frac{\alpha\beta}{C^2}\|Q_e\|^2 \lambda_{min}(M_e^p)^2 \Rightarrow \lambda_{min}(S_e) \geq \frac{\alpha\beta}{C^2} h_{min}^d$$

(ii) Q_e eigenvect corresponding to $\lambda_{max}(S_e)$ $\Rightarrow$

$$\lambda_{max}(S_e)\|Q_e\|^2 = S_e Q_e \cdot Q_e = \underbrace{A_e^{-1}B_e^T Q_e}_{= W_e} \cdot B_e^T Q_e$$

$$= W_e \cdot B_e^T Q$$

$$\leq \|B_e^T Q_e\|_* \|W_e\|_X \leq \frac{1}{\alpha}\|B_e^T Q_e\|_* \|A_e W_e\|_X$$

$\boxed{\text{definition of } \|\cdot\|_*}$ $\quad$ $\boxed{\|A_e u_e\|_* \geq \alpha\|u_e\|_X}$

$$\leq \frac{1}{\alpha}\|B_e^T Q_e\|_*^2 \leq \frac{C^2}{\alpha}\|Q_e\|_M^2 \leq \frac{C^2}{\alpha}\|Q_e\|^2 \lambda_{max}(M_e^p)^2$$

$$\Rightarrow \lambda_{max}(S_e) \leq \frac{C^2}{\alpha}\lambda_{max}(M_e^p)^2$$

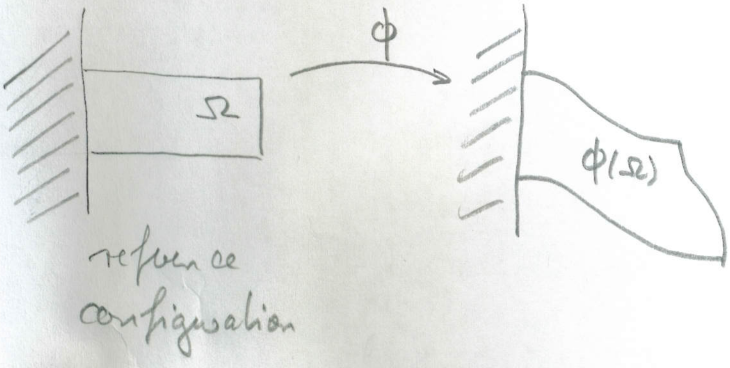
(i), (ii) $\Rightarrow$ $\boxed{\kappa(S_e) \leq C\,\kappa(M_e^p)}$ $\square$

consequence: on uniform meshes:

- the outer iteration (e.g. via GG) converges in $O(1)$ steps for fixed threshold.
- thus, even though the UZAWA scheme requires a nested iteration it is efficient

Next, we study elastic material with a focus on thin plates!

To this end, we briefly review volumetric elasticity:



ϕ deformation ($x^\phi = \phi(x)$)

$u(x) = \phi(x) - x$ displacement

$D\phi(x) \in \mathbb{R}^{d,d}$ deformation gradient

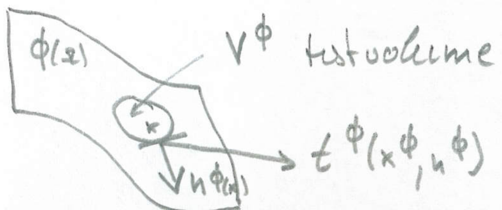
$$|\phi(x+y) - \phi(x)|^2 = |D\phi(x)y|^2 + O(|y|^3)$$
$$= \underbrace{D\phi(x)^T D\phi(x)}_{\text{Cauchy Green strain tensor}} y\cdot y + O(|y|^3)$$

(controls the local change of length)

$$E = \frac{1}{2}(D\phi^T D\phi - \mathbb{1}) = \frac{1}{2}((\mathbb{1}+Du^T)(\mathbb{1}+Du) - \mathbb{1})$$
$$= \underbrace{\frac{1}{2}(Du^T + Du)}_{=:\ \epsilon[u] \text{ linearized strain tensor}} + \frac{1}{2}Du^T Du$$

$$Du(x) = \epsilon[u] + \underbrace{\frac{Du(x) - Du(x)^T}{2}}_{\in\, so(d) \text{ generating infinitesimal rotations}}$$

stress balance (Cauchy-Euler axiom):



n^ϕ normal on ∂V^ϕ

$\exists\, t^\phi : \phi(\Omega) \times S^{d-1} \to \mathbb{R}^d$ with

$$\boxed{\int_{\partial V^\phi} t^\phi(x^\phi, n^\phi)\, da^\phi + \int_{V^\phi} f^\phi(x)\, dx^\phi = 0} \quad (*)$$

volume force density

(stress balance with t^ϕ force density per unit surface element)

$$\boxed{\int_{\partial V^\phi} x^\phi \wedge t^\phi(x^\phi, n^\phi)\, da^\phi + \int_{V^\phi} x^\phi \wedge f^\phi(x)\, dx^\phi = 0}$$

(conservation of angular momentum)

Cauchy thm: $t^\phi \in C^0 \Rightarrow t^\phi(x^\phi, n^\phi) = \underbrace{T^\phi(x^\phi)}\, n^\phi$

and $T^\phi = (T^\phi)^T$ stress tensor $\phi(\Omega) \to \mathbb{R}^{d,d}$

$t^\phi(\cdot, n) \in C^1 \xrightarrow[\text{Gauß thm}]{}$ $\boxed{-\operatorname{div}^\phi T^\phi(x^\phi) = f^\phi(x^\phi)}$ $\left(\operatorname{div} T = \left(\sum_j \partial_j T_{ij}\right)_i\right)$

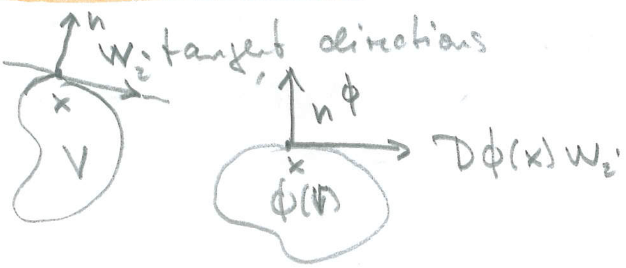
conservation law in deformed coordinates

Now, we transform this conservation law to the (easier to handle) reference configuration:

(i) transformation of forces: $V^\phi = \phi(V)$ $\quad \det D\phi > 0$

$$\int_{V^\phi} f^\phi(x^\phi)\, dx^\phi \underset{\text{trf thm}}{=} \int_V \underbrace{f^\phi(\phi(x))\, \det D\phi(x)}_{=:\, f(x)}\, dx$$

(ii) transformation of normals and area elements:



$n^\phi \cdot D\phi\, w_i = 0 \Rightarrow$

$D\phi^T n^\phi(\phi(x)) = 0 \Rightarrow$

$i = 1, \dots, d-1$

$\Rightarrow \quad D\phi^T n^\phi(\phi(x)) \parallel n(x)$

$\Rightarrow \quad n^\phi(\phi(x)) \parallel D\phi^{-T}(x)\, n(x)$

$\det D\phi(x) > 0$

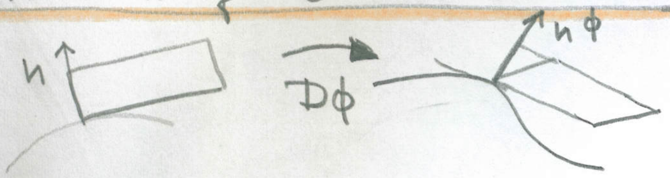
$\Rightarrow \quad n^\phi(\phi(x)) = \dfrac{D\phi^{-T}(x)\, n(x)}{|\text{---}\;"\;\text{---}|}$

Cramers rule: $A^{-1} = \det A \ \operatorname{cof} A^T \qquad (\operatorname{cof} A)_{ij} = \det(A_{ij})$

cancelation row i column j

$\Rightarrow \quad n^\phi(x^\phi) = \dfrac{\operatorname{cof} D\phi(x)\, n(x)}{|\text{---}\;"\;\text{---}|}$

deformation of the area element (surface transform):



$\Rightarrow$

$$da^\phi = \frac{\det D\phi}{D\phi\, n \cdot n^\phi}\, da = \frac{\det D\phi \ |\operatorname{cof} D\phi\, n|}{D\phi\, n \cdot D\phi^{-T} n \det D\phi}\, da = |\operatorname{cof} D\phi\, n|\, da$$

(iii) $$\int_{V^\phi} T^\phi n^\phi\, da^\phi = \int_V T^\phi \frac{\operatorname{cof} D\phi\, n}{|\operatorname{cof} D\phi\, n|} |\operatorname{cof} D\phi\, n|\, da$$

$$= \int_V \underbrace{T^\phi_{(\phi(x))} \operatorname{cof} D\phi(x)}_{=:\, T(x)} n(x)\, da$$

1st Piola-Kirchhoff stress tensor

(i) + (iii) $\overset{(*)}{\Rightarrow}$ $$\int_{\partial V} T n\, da + \int_V f\, dx = 0$$

$\Rightarrow$ $\boxed{-\operatorname{div} T(x) = f(x)}$

Gauß thm

T is not symmetric!

$\Sigma(x) := D\phi^{-1} T(x) = D\phi^{-1} T^\phi D\phi^{-T} \det D\phi$ symmetric

(2nd Piola-Kirchhoff stress tensor)

$\Rightarrow \quad -\operatorname{div}\left(D\phi(x)\, \Sigma(x) \right) = f(x)$

<u>material law of elasticity</u>: $T(x) = \hat{T}(x, D\phi(x))$

$\uparrow$ possibly non homogeneous

<u>Now assume homogeneous material</u>:

$T(x) = \hat{T}(D\phi(x)) \quad , \quad \Sigma(x) = \hat{\Sigma}(D\phi(x))$

Ⓡ <u>rigid body motion invariance</u>

$Q\, t(x,n) = t(Qx, Qn) \quad \forall x \in \mathbb{R}^3 \ \forall n \in S^2 \ \forall Q \in SO(3)$

$\Rightarrow \ Q\, T(x)\, Q^T = T(Qx) \ \Rightarrow \ \hat{T}(QF) = Q\, \hat{T}(F)\, Q^T$

One can show: $\quad \hat{\Sigma}(QF) = \hat{\Sigma}(F) = \bar{\Sigma}(F^T F)$

Ⓘ <u>isotropy</u> $\quad \hat{T}(F) = \hat{T}(FQ) \quad \forall Q \in SO(3)$

polar decomposition:

$F \in \mathbb{R}^{d,d}$ can be decomposed $\ F = BQ$

with $B \in \mathbb{R}^{d,d}$ symmetric, $Q \in SO(d)$, i.e. $B = \sqrt{FF^T}$

$\Rightarrow \ \hat{T}(F) = \hat{T}\left(\sqrt{FF^T}\, Q\right) = \hat{T}\left(\sqrt{FF^T}\right) =: \bar{T}(FF^T)$

<u>Rivlin-Erichsen representation thm</u>

Ⓡ + Ⓘ $\Rightarrow \quad \bar{\Sigma}(C) = \gamma_0 \mathbb{1} + \gamma_1 C + \gamma_2 C^2, \quad C = F^T F$

with $\gamma_0, \gamma_1, \gamma_2$ functions of the invariants

$\boxed{d=3} \qquad i_C = (\operatorname{tr} C, \ \operatorname{tr} \operatorname{cof} C, \ \det C)$

Linear material law

$$\bar{\Sigma}(C) = \bar{\Sigma}(\mathbb{1} + 2E) = \lambda \operatorname{tr} E \cdot \mathbb{1} + 2\mu E,$$

(St. Venant - Kirchhoff material)

Geometric linearization:

$$E = \tfrac{1}{2}(D\phi^T D\phi - \mathbb{1}) \rightsquigarrow \varepsilon = \varepsilon[u], \quad u = \phi - id$$

$$\sigma = \lambda \operatorname{tr} \varepsilon[u] \cdot \mathbb{1} + 2\mu\, \varepsilon[u] \quad \text{(linearized stress tensor)}$$

$\lambda, \mu \geq 0$ Lamé constants.

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \text{(elasticity modulus)}, \quad E > 0$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad \text{(Poisson ratio)}, \quad 0 < \nu < \tfrac{1}{2}$$

almost incompressible material: $\lambda \gg \mu$.

Add boundary conditions:

$\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, $\Gamma_D \cap \Gamma_N = \emptyset$

- $\phi = id \iff u = 0$ on Γ_D (Dirichlet b.c.)
- $\hat{T}(D\phi) \cdot n = g \iff \sigma \cdot n = g$ on Γ_N (Neumann b.c.)

$$(P_{elast}) \left[\begin{aligned} -\operatorname{div} \hat{T}(x, D\phi) &= f && \text{in } \Omega \\ \phi &= id && \text{on } \Gamma_D \\ \hat{T}(x, D\phi) \cdot n &= g && \text{on } \Gamma_N \end{aligned}\right]$$

$$(P^{lin}_{elast}) \left[\begin{aligned} -\operatorname{div} \sigma &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \sigma \cdot n &= g && \text{on } \Gamma_N \end{aligned}\right]$$

Towards a variational formulation:

Hyperelasticity: $\exists$ functional $\hat{W} : \Omega \times \mathbb{R}^{d,d} \to \mathbb{R}$

$$\hat{T}(x, F) = \frac{\partial \hat{W}}{\partial F}(x, F) \qquad \forall x \in \Omega, \ F \in \mathbb{R}^{d,d}$$

stored energy: $\mathcal{W}[\phi] = \int_\Omega \hat{W}(x, D\phi(x))\, dx$.

(R) $\leadsto$ $\hat{W}(x, F) = \overline{W}(x, F^T F)$

$$\overline{\Sigma}(x, C) = 2\, \frac{\partial \overline{W}(x, C)}{\partial C}$$

St. V.-K.: $\hat{W}^{VK}(x, F) = \frac{\lambda}{2}(\operatorname{tr} E)^2 + \mu\, E : E$

Linear: $\hat{W}^{lin}(x, \mathbb{1} + Du) = \frac{\lambda}{2}(\operatorname{tr} \varepsilon[u])^2 + \mu\, \varepsilon[u] : \varepsilon[u]$

$\boxed{d=3}$ $\quad \sigma = \lambda \operatorname{tr} \varepsilon[u]\, \underline{\mathbb{1}} + 2\mu\, \varepsilon[u]$

$$= \frac{E}{1+\nu}\left(\varepsilon[u] + \frac{\nu}{1-2\nu} \operatorname{tr} \varepsilon[u]\, \underline{\mathbb{1}} \right) =: \underset{\substack{\uparrow \\ \text{elasticity tensor} \\ \text{(fourth order tensor)}}}{C}\, \varepsilon[u]$$

Voigt notation: (σ, ε symmetric $\to$ 6 entries)

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & 1-2\nu & & \\ & & & & 1-2\nu & \\ & & & & & 1-2\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{pmatrix}$$

$\Rightarrow \hat{W}^{lin}(x, \mathbb{1} + Du) = \frac{1}{2} \sigma : \varepsilon[u] = \frac{1}{2} C\, \varepsilon[u] : \varepsilon[u]$

$$\Rightarrow \mathcal{W}^{lin}[u] := \int_\Omega \frac{1}{2} C\, \underbrace{\varepsilon[u] : \varepsilon[u]}_{\underset{\text{symmetry}}{=} Du : Du}\, dx$$

Variational formulation:

$$(E^{lin}_{elast}) \quad \mathcal{E}[u] = \mathcal{W}^{lin}[u] - \int_\Omega f u \, dx - \int_{\Gamma_N} g u \, da$$

4.18 Lemma (Korn's inequality)

Let Ω be bounded & simply connected, $\partial\Omega$ Lipschitz.

Then $\int_\Omega \varepsilon[u] : \varepsilon[u] + u^2 \, dx \geq C \|u\|^2_{1,2,\Omega}$

If $\mathcal{H}^{d-1}(\Gamma_D) > 0$: $\int_\Omega \varepsilon[u] : \varepsilon[u] \, dx \geq C \|u\|^2_{1,2,\Omega} \quad \forall u \in H^{1,2}_{\Gamma_D}(\Omega)$.

($H^{1,2}_\Gamma(\Omega)$ is closure of $\{u \in C^\infty(\Omega)^3 : u(x) = 0 \ \forall x \in \Gamma_D\}$ w.r.t. $\|\cdot\|_{1,2,\Omega}$).

4.19 Thm (Existence, uniqueness)

Let Ω be bounded & simply connected, $\partial\Omega$ Lipschitz,

$\overline{\Gamma_D} \cup \overline{\Gamma_N} = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, linear St. Venant-Kirchhoff.

Let $\mathcal{H}^{d-1}(\Gamma_D) > 0$, $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$.

Then $\exists !$ minimizer u of (E^{lin}_{elast}), which is given as the solution of (P^{lin}_{elast}).

Proof: Korn $\rightsquigarrow$ coerciveness; apply Lax-Milgram $\square$

$\left(a(u,v) := \frac{1}{2} \int_\Omega C \varepsilon[u] : \varepsilon[v] \, dx \right)$

Remark: Three "elastic" quantities: $\begin{cases} \sigma & \text{stress tensor} \\ \varepsilon & \text{linear strain tensor} \\ u & \text{displacement} \end{cases}$

We used $\sigma = C\varepsilon[u]$, $\varepsilon[u] = \frac{1}{2}(Du^T + Du)$.

$\rightsquigarrow$ solved PDE in u only.

Alternative: mixed method with variables u and σ

Consistency term: $\sigma - C\varepsilon[u] \stackrel{!}{=} 0$ (Hellinger-Reissner)

<u>1st approach:</u> $\mathbb{X} = (L^2(\Omega))^{d^2}$, $M = (H^{1,2}_{\Gamma_D}(\Omega))^d$

$$a(\sigma,\tau) = (C^{-1}\sigma,\tau)_{0,2,\Omega} \qquad b(\tau,v) = -(\tau, \varepsilon[v])_{0,2,\Omega}$$

<u>2nd approach:</u> $\mathbb{X} = H(\mathrm{div},\Omega) \subset (L^2(\Omega))^{d^2}$, $M = (L^2(\Omega))^d$

$$a(\sigma,\tau) = (C^{-1}\sigma,\tau)_{0,2,\Omega} \qquad b(\tau,v) = (\mathrm{div}\,\tau, v)_{0,2,\Omega}$$

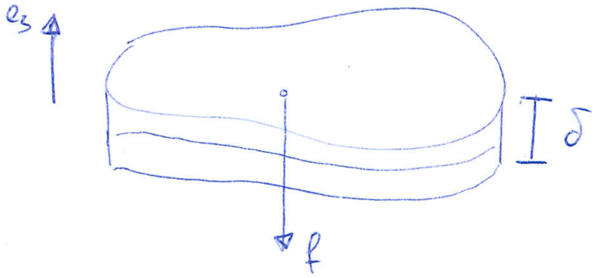
$\rightsquigarrow$ compare to mixed methods in context of Poisson pb.

Now we investigate plates:

$\Omega = \Omega_\delta = \omega \times (-\frac{\delta}{2}, \frac{\delta}{2})$, $0 < \delta \ll 1$.

where $\omega \subset \mathbb{R}^2$ bounded & simply connected, $\partial\omega$ Lipschitz.

<u>Notation:</u> $x = (x_1, x_2, x_3) = (x', x_3)$



forces $f \perp \mathbb{R}^2 \times \{0\}$ independent of x_3

$f(x) = f(x')$.

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} : \Omega_h \longrightarrow \mathbb{R}^3$

<u>Assumptions:</u> (H1) normal segments undergo affine transformations

(H2) displ. in vertical direction only depends on x', i.e. $u_3(x) = w(x')$

(H3) points on $\omega \times \{0\}$ are only deformed in e_3-direction

(H1)-(H3) $\quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}(x) = -x_3\, \Theta(x')$

$u_3(x) = w(x')$ (transversal displacement, bending)

(H4) normal stress vanishes: $\sigma_{33} = 0$

<u>Remark</u>: (H1)-(H4): Reissner-Mindlin plate model

<u>additional assumption</u>:

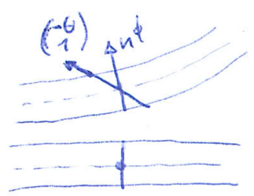
(H5) normal hypothesis (Kirchhoff-Love)

deformed normal is again normal to deformed midsurface.

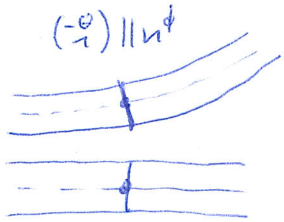
$$\phi(x) = x + u(x) \Rightarrow \begin{array}{l} \partial_i \phi(x',0) = \begin{pmatrix} e_i \\ \partial_i w(x') \end{pmatrix}, \quad i=1,2 \\ \partial_3 \phi(x',0) = \begin{pmatrix} -\theta(x') \\ 1 \end{pmatrix} \end{array}$$

$$(H5) \Rightarrow \partial_i \phi(x',0) \perp \partial_3 \phi(x',0) \Rightarrow \partial_i w(x') = \theta_i(x'), \quad i=1,2$$

$$\Leftrightarrow \nabla w = \theta.$$



(Mindlin-Reissner)



(Kirchhoff-Love)

$$u = \begin{pmatrix} -x_3 \theta(x') \\ w(x') \end{pmatrix} \Rightarrow Du = \left(\begin{array}{c|c} -x_3 D_{x'}\theta & -\theta \\ \hline \nabla_{x'} w^T & 0 \end{array}\right)$$

$$\Rightarrow \varepsilon[u] = \left(\begin{array}{c|c} -x_3\, \varepsilon[\theta] & \frac{1}{2}(\nabla_{x'} w - \theta) \\ \hline \frac{1}{2}(\nabla_{x'} w - \theta)^T & 0 \end{array}\right), \quad \varepsilon[\theta] = \frac{1}{2}\left(D_{x'}\theta^T + D_{x'}\theta\right)$$

$$\sigma = \frac{E}{1+\nu}\left(\varepsilon[u] + \frac{\nu}{1-2\nu} \operatorname{tr} \varepsilon[u] \, \mathbb{1} \right)$$

$$\sigma : \varepsilon[u] \underset{(114)}{=} \frac{E}{1+\nu}\left(2\varepsilon_{12}^2 + 2\varepsilon_{13}^2 + 2\varepsilon_{23}^2 + \frac{1-\nu}{1-2\nu}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right) + \frac{2\nu}{1-2\nu} \varepsilon_{11}\varepsilon_{22} \right)$$

$$= \frac{E}{1+\nu}\left(\sum_{i,j=1}^{2} \varepsilon_{ij}^2 + \frac{\nu}{1-2\nu}\left(\varepsilon_{11}^2 + \varepsilon_{22}^2\right) + \frac{2\nu}{1-2\nu}\left(\varepsilon_{11}\varepsilon_{22}\right) + 2\sum_{j=1}^{2} \varepsilon_{j3} \right)$$

$$= \frac{E}{1+\nu}\left(\sum_{\substack{i,j=1 \\ (i,j)\neq(3,3)}}^{3} \varepsilon_{ij}^2 + \frac{\nu}{1-2\nu}\left(\varepsilon_{11} + \varepsilon_{22}\right)^2 \right)$$

$$\mathcal{E}[u] = \mathcal{E}^{MR}[w,\theta) = \int_\Omega \frac{1}{2}\sigma : \varepsilon[u] - f(x)\cdot u \, dx$$

$$= \frac{1}{2}\int_\omega \int_{-\delta/2}^{\delta/2} \frac{E}{1+\nu} x_3^2 \left(\varepsilon[\theta] : \varepsilon[\theta] + \frac{\nu}{1-2\nu}(\operatorname{div}\theta)^2 \right) dx_3 \, dx'$$

$$+ \frac{1}{2}\int_\omega \int_{-\delta/2}^{\delta/2} \frac{E}{1+\nu} \left| \nabla_{x'} w - \theta \right|^2 - f(x') w \, dx_3 \, dx'$$

$$= \delta^3 \frac{E}{24(1+\nu)} \int_\omega \varepsilon[\theta] : \varepsilon[\theta] + \frac{\nu}{1-2\nu}(\operatorname{div}\theta)^2 \, dx'$$

$$+ \delta \frac{E}{2(1+\nu)} \int_\omega \left| \nabla_{x'} w - \theta \right|^2 - f w \, dx'$$

Define: $$a(\theta,\psi) := \frac{E}{12(1+\nu)} \int_\omega \varepsilon[\theta] : \varepsilon[\psi] + \frac{\nu}{1-2\nu} \operatorname{div}\theta \cdot \operatorname{div}\psi \, dx'$$

$$\gamma := \frac{E}{2(1+\nu)}$$

$$\Longrightarrow \mathcal{E}^{MR}[w,\theta] = \frac{\delta^3}{2} a(\theta,\theta) + \delta\left(\gamma \left\| \nabla w - \theta \right\|^2 - (f,w)_{0,2,\omega} \right)$$

Euler-Lagrange equation:

$$(P^{MR})\begin{cases} \text{Find } (\theta,w)\in H_0^1(\omega)\times H_0^1(\omega) \text{ s.t.} \\ \delta^3 a(\theta,\psi) + 2\delta\gamma\,(\theta-\nabla w,\psi)_{0,2,\omega} = 0 \quad \forall\psi\in H_0^1(\omega) \\ 2\delta\gamma\,(\nabla w,\nabla \vartheta)_{0,2,\omega} = (f,\vartheta)_{0,2,\omega} \quad \forall\vartheta\in H_0^1(\omega) \end{cases}$$

Now consider Kirchhoff-Love:

$$\left.\begin{array}{l} \theta=\nabla w,\ \mathcal{E}[\theta]=D^2w \\ \psi=\nabla v,\ \mathcal{E}[\psi]=D^2 v \end{array}\right\} \Rightarrow a(\theta,\psi) = \frac{E}{12(1+\nu)}\int_\omega \left(D^2w : D^2v + \frac{\nu}{1-2\nu}\Delta w\,\Delta v\right)dx'$$

$$\mathcal{E}^{KL}[w] = \frac{\delta^3}{2}a(\nabla w,\nabla w) - \delta\,(f,w)_{0,2,\omega}$$

Euler-Lagrange equation: Find $w\in H_0^2(\omega)$ s.t.

$$\delta^3 a(\nabla w,\nabla v) = \delta(f,v)_{0,2,\omega} \quad \forall v\in H_0^2(\omega)$$

Remark: standard FE approach requires C^1-elements.

$\rightsquigarrow$ mixed method, i.e. saddle point formulation!

Ansatz: Minimize $\frac{1}{2}a(\theta,\theta)-(f,w)_{0,2,\omega}$,

subject to the constraint $\theta=\nabla w$.

$\boxed{\text{Set } \delta=1}$

$$(P_S^{KL})\begin{cases} \text{Find } (w,\theta)\in \underline{X} \text{ and } \lambda\in M, \text{ s.t.} \\ a(\theta,\psi) + (\nabla v-\psi,\lambda)_{0,2,\omega} = (f,v)_{0,2,\omega} \quad \forall (v,\psi)\in\underline{X} \\ (\nabla w-\theta,\mu)_{0,2,\omega} = 0 \quad \forall\mu\in M \end{cases}$$

First ansatz for spaces X, M:

$$X = H_0^2(\omega) \times (H_0^1(\omega))^2, \quad M = H^{-1}(\omega)$$

$$b((w,\theta),\mu) := (\nabla w - \theta, \mu)_{0,2,\omega}$$

- $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ bounded ($\checkmark$)
- V-ellipticity with $V = \operatorname{Ker} B$.

$$a(\psi,\psi) \underset{\text{Korn's inequality}}{\geq} c\,\|\psi\|_{1,2}^2 \underset{\substack{\psi=\nabla v\\ \text{on Ker } B}}{=} \frac{c}{2}\left(\|\psi\|_{1,2}^2 + \|\nabla v\|_{1,2}^2\right)$$

$$\underset{\substack{\text{Poincaré}\\ +\text{ b.c.}}}{\geq} c_1\left(\|\psi\|_{1,2}^2 + \|v\|_{2,2}^2\right)$$

($\checkmark$)

- $$\sup_{(v,\psi)\in H_0^2\times(H_0^1)^2} \frac{(\nabla v - \psi, \eta)_{0,2,\omega}}{\|v\|_{2,2} + \|\psi\|_{1,2}} \underset{[v=0]}{\geq} \sup_{\psi\in(H_0^1)^2} \frac{-(\psi,\eta)}{\|\psi\|_{1,2}} = \|\eta\|_{-1,2}$$

$\Rightarrow$ $b(\cdot,\cdot)$ fulfills the inf-sup condition ($\checkmark$)

$\rightsquigarrow$ Existence of a saddle point $((w,\theta),\lambda)$ with

$$\|w\|_{2,2,\omega} + \|\theta\|_{1,2,\omega} + \|\lambda\|_{-1,2,\omega} \leq C\,\|f\|_{-1,2,\omega}$$

Alternative ansatz: (with less regularity)

$$X = (H_0^1(\omega))^3, \quad M = H^{-1}(\operatorname{div},\omega)$$

$$H^{-1}(\operatorname{div},\omega) = \overline{C^\infty(\omega)^2}^{\|\cdot\|_{H^{-1}(\operatorname{div},\omega)}}, \quad \|\eta\|_{H^{-1}(\operatorname{div},\omega)}^2 = \|\eta\|_{-1,2,\omega}^2 + \|\operatorname{div}\eta\|_{-1,2,\omega}^2$$

Remark: Under the assumption that $\partial\omega$ is piecewise smooth:

$$H^{-1}(\operatorname{div},\omega) = \{\eta \in H^{-1}(\omega) : \operatorname{div}\eta \in H^{-1}(\omega)\}$$

$\boxed{\text{boundedness}}$ $a(\cdot,\cdot)$ ($\checkmark$)

$$b((v,\psi),\mu) = (\nabla v - \psi, \mu)_{0,2} = -(v, \operatorname{div}\mu)_{0,2} - (\psi,\mu)_{0,2}$$

$$\leq \|v\|_{1,2} \|\operatorname{div}\mu\|_{-1,2} + \|\psi\|_{1,2} \|\mu\|_{-1,2}$$

$$\leq C \, \|(v,\psi)\|_{X} \cdot \|\mu\|_{M} \qquad (\checkmark)$$

$\boxed{\text{V-ellipticity}}$

$$a(\psi,\psi) \underset{\text{as above}}{\geq} \frac{c}{2}\left(\|\psi\|_{1,2}^2 + \|\nabla v\|_{1,2}^2\right) \underset{\text{Poincaré}}{\geq} C_1 \|(\psi,v)\|_{1,2}^2$$

($\checkmark$)

$\boxed{\text{inf-sup condition}}$

$$\sup_{v,\psi} \frac{(\nabla v - \psi,\mu)_{0,2}}{\left(\|v\|_{1,2}^2 + \|\psi\|_{1,2}^2\right)^{1/2}} \geq \frac{1}{2}\left(\sup_{\psi} \frac{-(\psi,\mu)_{0,2}}{\|\psi\|_{1,2}} + \sup_{v} \frac{\overbrace{(\nabla v,\mu)_{0,2}}^{-(v,\operatorname{div}\mu)_{0,2}}}{\|v\|_{1,2}} \right)$$

$$= \frac{1}{2}\left(\|\mu\|_{-1,2} + \|\operatorname{div}\mu\|_{-1,2} \right) \qquad (\checkmark)$$

$\rightsquigarrow$ Existence of a saddle point $((w,\theta),\lambda) \in X \times M$ with

$$\|w\|_{1,2,\omega} + \|\theta\|_{1,2,\omega} + \|\lambda\|_{H^{-1}(\operatorname{div},\omega)} \leq C \|f\|_{-1,2,\omega}$$

By regularity theory (ω convex):

$$\|w\|_{3,2,\omega} + \|\theta\|_{2,2,\omega} + \|\lambda\|_{0,2,\omega} \leq C \|f\|_{-1,2,\omega}$$