Sheet 0

Submission on **-**.

For completion, we state the divergence theorem, also known as Gauss' theorem. On this sheet, let $\Omega \subset \mathbb{R}^d$ be an open and bounded subset, with smooth boundary $\partial\Omega$ and outer normal vector $n(x)$.

Theorem. For a continuously differentiable vector field $F\colon \Omega \longrightarrow \mathbb{R}^d$, it holds

$$\int_\Omega \operatorname{div} F(x)\,\mathrm{d}x = \int_{\partial\Omega} F(x)\cdot n(x)\,\mathrm{d}x\,.$$

Exercise 1. (Green's identity)

Prove Green's first identity: Let $u, v \in \mathcal{C}^2(\Omega)$. Then one has

$$\int_\Omega v(x)\Delta u(x)\,\mathrm{d}x = -\int_\Omega \nabla v(x)\cdot\nabla u(x)\,\mathrm{d}x + \int_{\partial\Omega} v(x)\nabla u(x)\cdot n(x)\,\mathrm{d}x\,.$$

(0 points)

Exercise 2. (maximum principle)

Let $\Omega = \mathbb{R}^d$. Let $A(x)$ be a continuously differentiable, symmetric, uniformly positive definite matrix and consider the operator

$$L_x(u)(x) := -\operatorname{div}[A(x)\nabla u(x)]$$

for functions $u \in \mathcal{C}^2(\Omega)$. We say that L_x is an elliptic operator in divergence form. We first consider how some differential operators behave under a change of variable.

a) For an invertible matrix $B \in \mathbb{R}^{d\times d}$, we consider the coordinate transform $x = By$ and the scalar functions $v(y) = u(By) = u(x)$. State $\nabla_y v(y)$ in terms of B, u and x.

b) For two vector fields V, U on Ω with $V(y) = U(By) = U(x)$, state $\operatorname{div}_y V(y)$ in terms of B, U and x.

c) For the scalar functions $v(y) = u(By) = u(x)$, use a) and b) to state $\operatorname{div}_y[A(By)\nabla_y v(y)]$ in terms of B, u, A, x. Define an operator L_y such that $L_y(v)(y) = L_x(u)(x)$ for all u, v with $v(y) = u(By) = u(x)$.

d) Show that a function u with $L_x(u)(x) \leq 0$ for all $x \in \Omega$ has no strict local maximum.

(0 points)