Scientific Computing 1

Winter term 2017/18
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Sheet 1

Submission on **Thursday, 26.10.17**.

Exercise 1. (Helmholtz equation in 1D)

Let $I = (0, 1)$ be the unit interval and consider the Eigenfunctions to the onedimensional Laplace operator, i.e. functions $u \in \mathcal{C}^2(I) \cap \mathcal{C}(\bar{I})$ that satisfy

$$\begin{aligned} -u''(x) &= \lambda u(x) \text{ in } I\,, \\ u(0) &= 0\,, \\ u(1) &= 0 \end{aligned}$$

for some $\lambda \in \mathbb{R}$.

a) Find Eigenfunctions u_λ and the corresponding Eigenvalues $\lambda \in \mathbb{R}$.

b) Show that Eigenfunctions with different Eigenvalues are orthogonal with respect to the inner product $(u, v)_{L^2(I)} = \int_I u(x)v(x)\,\mathrm{d}x$.

(4 points)

Exercise 2. (discrete Helmholtz equation in 1D)

Let $I = (0, 1)$ be the unit interval and $x_i = i/n$, $i = 0, \dots, n$ be the standard partition. Use the second order central difference scheme

$$u''(x_i) \approx n^2\Big(u(x_{i-1}) - 2u(x_i) + u(x_{i+1})\Big)\,, \quad i = 1, \dots, n-1$$

to discretize the 1D Helmholtz equation from Exercise 1. Show that the discrete problem has only nonnegative Eigenvalues.

(4 points)

Exercise 3. (condition number)

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{d \times d}\,.$$

a) Show that A has the Eigenvectors $v_k = \left(\sin\left(\frac{1\pi k}{d+1}\right), \dots, \sin\left(\frac{d\pi k}{d+1}\right)\right)^T \in \mathbb{R}^d$ and compute the corresponding Eigenvalues λ_k.

b) Study the condition number $\kappa(A) = \lambda_{\max}(A)/\lambda_{\min}(A)$ for increasing d. Show that it grows like $\mathcal{O}(d^2)$.

(4 points)

Exercise 4. (Laplacian matrix)

Let $G = (V, E)$ be a finite, simple and connected graph with vertices $V = \{v_1, \dots, v_d\}$. The Laplacian matrix $L \in \mathbb{R}^{d \times d}$ associated to G is given by

$$L_{ij} = \begin{cases} \deg(v_i) & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\deg(v)$ is defined as the number of adjacent vertices of v in V.

a) State two finite, simple and connected graphs with 4 vertices of your choice and calculated the associated Laplacian matrix L.

b) We assign to every vertex v_i a value $x_i \in \mathbb{R}$. Let the component vector $x \in \mathbb{R}^d$ satisfy $(Lx)_i \leq 0$ for $i = 1, \dots, d$. Show that x is constant, i.e. $x_i = x_j$ for $i, j = 1, \dots, d$.

(4 points)