Scientific Computing 1

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Sheet 10

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Exercise 1. (The Stokes Equation)

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded and convex domain with smooth boundary $\partial\Omega$ and normal unit vector $n\colon \partial\Omega \to \mathbb{R}^n$. The motion of an incompressible viscous fluid with velocity field $u\colon \Omega \to \mathbb{R}^n$ can be modeled with the PDE

$$\begin{aligned} \Delta u + \nabla p &= -f && \text{in } \Omega\,, \\ \operatorname{div} u &= 0 && \text{in } \Omega\,, \\ u &= u_0 && \text{on } \partial\Omega\,. \end{aligned}$$

Here, $p\colon \Omega \to \mathbb{R}$ is the pressure density, $f\colon \Omega \to \mathbb{R}^n$ is an external force field and Δ is the componentwise Laplacian.

a) Assume that a (strong) solution (u, p) exists. Show that the boundary condition u_0 must satisfy

$$\int_{\partial\Omega} u_0 \cdot n \,\mathrm{d}S = 0\,.$$

From now on, we consider homogeneous boundary conditions $u_0 = 0$. Moreover, for a solution (u, p), p is only determined up to an additive constant, so we additionally enforce

$$\int_{\Omega} p \,\mathrm{d}x = 0\,.$$

We proceed to the weak formulation: Define

$$X = H_0^1(\Omega)^n, \quad M = \left\{ q \in L^2(\Omega)\colon \int_\Omega q \,\mathrm{d}x = 0 \right\}\,,$$

$$\begin{aligned} a(u,v) &= \int_\Omega \operatorname{Tr}\big[(Du)^\top (Dv)\big] \,\mathrm{d}x\,, \\ b(v,q) &= \int_\Omega q \operatorname{div} v \,\mathrm{d}x \end{aligned}$$

with $Du = [\partial_j u_i]_{i,j=1}^n$ the Jacobian matrix of u. We get a saddle point problem:
Find $(u, p) \in X \times M$ such that

$$\begin{aligned} a(u,v) + b(v,p) &= (f,v)_{L^2(\Omega)} \\ b(u,q) &= 0 \end{aligned}$$

is satisfied for all $v \in X$, $q \in M$.

b) Show that $a(u, v)$ is elliptic on the whole space X, with respect to the $H^1(\Omega)^n$-norm

$$\|u\|_{1,n} = \left(\sum_{i=1}^n \|u_i\|_{H^1(\Omega)}^2 \right)^{1/2}\,.$$

c) The form $b(v,q)$ induces an operator $B'\colon M \to X'$ via

$$(v, B'q)_{X,X'} = b(v,q)\,.$$

Show that B' restricted to $M \cap H^1(\Omega)$ reduces to the negative weak gradient operator $q \mapsto -\nabla q \in L^2(\Omega)^n \subset X'$.

(8 points)

Exercise 2. (instable discretization)

We consider a finite element discretization of the Stokes equation for $\Omega = [0,1]^2$ with the standard rectangular mesh $\mathcal{T}_h = \{T_{ij}\}_{i,j=0}^{m-1}$, where $h = 1/m$ and

$$T_{ij} = [ih, (i+1)h] \times [jh, (j+1)h]$$

for $i,j = 0,\dots,m$. Moreover, we choose piecewise continuous, bilinear elements for the velocity and piecewise discontinuous, constant elements for the pressure:

$$\begin{aligned} X_h &= \{v \in X\colon v_1|_T, v_2|_T \in Q(T) \text{ for } T \in \mathcal{T}_h\}\,,\\ M_h &= \{q \in M\colon q|_T \text{ constant for } T \in \mathcal{T}_h\}\,. \end{aligned}$$

Note that a function $v \in X_h$ is completely determined by its values at the nodes (ih, jh) for $i,j = 1,\dots,m-1$. To simplify notation, we set $v = (v_1, v_2) = (u, w)$ and $u_{ij} = u(ih, jh)$, $w_{ij} = w(ih, jh)$, as well as $q_{i+1/2,\,j+1/2} = q((i+1/2)h, (j+1/2)h)$.

a) Show that for $v \in X_h$, $q \in M_h$ one has

$$\int_\Omega q \operatorname{div} v \,\mathrm{d}x = h^2 \sum_{i,j=1}^{m-1} \left[u_{ij}(\nabla_1 q)_{ij} + w_{ij}(\nabla_2 q)_{ij}\right],$$

with difference quotients

$$\begin{aligned} (\nabla_1 q)_{ij} &= \frac{1}{2h}\left[q_{i+1/2,\,j+1/2} + q_{i+1/2,\,j-1/2} - q_{i-1/2,\,j+1/2} - q_{i-1/2,\,j-1/2}\right],\\ (\nabla_2 q)_{ij} &= \frac{1}{2h}\left[q_{i+1/2,\,j+1/2} - q_{i+1/2,\,j-1/2} + q_{i-1/2,\,j+1/2} - q_{i-1/2,\,j-1/2}\right]. \end{aligned}$$

b) Show that the kernel of the operator $B_h'\colon M_h \to X_h'$ defined via

$$(v, B_h' q)_{X_h, X_h'} = \int_\Omega q \operatorname{div} v \,\mathrm{d}x$$

is

$$\ker B_h' = \operatorname{span}\{q^*\}$$

with

$$q^*_{i+1/2,\,j+1/2} = \begin{cases} 1 & i+j \text{ odd},\\ -1 & i+j \text{ even}. \end{cases}$$

We replace M_h with the reduced space $R_h = \{q \in M_h\colon (q, q^*)_{L^2(\Omega)} = 0\}$, to obtain injectivity of B_h' on R_h. However, this is still not enough to obtain stability in the limit $h \to 0$.

c) (Bonus exercise)
Show that there exists a constant $C > 0$ such that

$$\inf_{q \in R_h} \sup_{v \in X_h} \frac{b(v,q)}{\|v\|_{1,2}\|q\|_{L^2(\Omega)}} \le Ch\,.$$

(8+4 points)