Scientific Computing 1

Winter term 2017/18
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Sheet 2

Submission on **Thursday, 2.11.17**.

Exercise 1. (integrability)

Let $f\colon \mathbb{R}^d \longrightarrow \mathbb{R}$ be given by

$$f(x) = \|x\|_2^{-\alpha}$$

for $\alpha > 0$. Compute the range of all α for which f belongs to $L^1_{\text{loc}}(\mathbb{R}^d)$.

(4 points)

Exercise 2. (weak differentiability)

We consider the function $f \in L^1_{\text{loc}}(\mathbb{R})$, $f(x) = |x|$.

a) Show that f is weakly differentiable and compute its weak derivative.

b) Show that f is not twice weakly differentiable.

(4 points)

Exercise 3. (Helmholtz equation in 2D)

Let $\Omega = (0,1)^2$ be the unit square and consider the 2D Helmholtz equation

$$\begin{aligned} -\Delta u(x) &= \lambda u(x) \text{ in } \Omega \\ u(x) &= 0 \text{ on } \partial\Omega \end{aligned}$$

for functions $u \in \mathcal{C}^2(\Omega)$.

a) Let $u_1, u_2 \in \mathcal{C}^2(0,1)$ be solutions to the Helmholtz equation in 1D, with respective Eigenvalues λ_1, λ_2. Show that $u(x,y) = u_1(x)u_2(y)$ solves the 2D Helmholtz equation and compute the corresponding Eigenvalue.

b) We discretize the 2D Helmholtz equation with a regular grid $\{x_{i,j} = (i/n, j/n) \mid i,j = 1,\ldots,n-1\}$ and the five-point stencil

$$-\Delta u(x_{i,j}) \approx n^2\big(4u(x_{i,j}) - u(x_{i-1,j}) - u(x_{i,j-1}) - u(x_{i+1,j}) - u(x_{i,j+1})\big)\,.$$

State the resulting discrete system of $N = (n-1)^2$ equations. Here, use the ordering given by

$$U = (U_1, \ldots, U_N)^\top\,, \quad u(x_{i,j}) = U_{(n-1)(i-1)+j}\,.$$

(4 points)

Exercise 4. (Kronecker product)

Let $A, B \in \mathbb{R}^{d\times d}$ be two square matrices. We define the Kronecker product

$$A \otimes B = \begin{bmatrix} A_{11}B & \cdots & A_{1d}B \\ \vdots & \ddots & \vdots \\ A_{d1}B & \cdots & A_{dd}B \end{bmatrix} \in \mathbb{R}^{d^2\times d^2}$$

as a block matrix with $d \times d$ blocks $A_{ij}B$. We also define

$$u \otimes v = \begin{bmatrix} u_1 v \\ \vdots \\ u_d v \end{bmatrix} \in \mathbb{R}^{d^2}.$$

for vectors $u, v \in \mathbb{R}^d$.

a) Show that $(A \otimes B)(u \otimes v) = (Au) \otimes (Bv)$.

b) Let I be the d-dimensional identity matrix and define

$$L = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{d\times d}.$$

State the matrix $A = L \otimes I + I \otimes L$ explicitely. Find all Eigenvectors and Eigenvalues of A.

(4 points)