Scientific Computing 1

Winter term 2017/18
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Sheet 4

Submission on **Thursday, 16.11.17**.

Exercise 1. (piecewise linear finite elements in 1D)

Let $I = [0, 1]$ be the unit interval. We divide I into n subintervals $[x_i, x_{i+1}]$ with $x_i = i/n$, $i = 0, \dots, n$. Consider the piecewise linear nodal basis $\{\phi_1, \dots, \phi_{n-1}\}$ which is uniquely defined by $\phi_k(x_j) = \delta_{kj}$ for $k = 1, \dots, n-1$ and $j = 0, \dots, n$.

a) For the bilinear form $a(u, v) = \int_I u'(x)v'(x)\,\mathrm{d}x$, compute $a(\phi_i, \phi_j)$ for $i, j = 1, \dots, n-1$.

b) Consider the weak formulation of the 1D Poisson problem with homogeneous Dirichlet boundary conditions:
Find $u \in H_0^1(I)$ s.t.

$$a(u, v) = \int_I f(x)v(x)\,\mathrm{d}x \quad \forall v \in H_0^1(I)$$

for some $f \in L^2(I)$. Define $V_n = \operatorname{span}\{\phi_1, \dots, \phi_{n-1}\} \subset H_0^1(I)$. Use the Galerkin method to discretize the weak formulation with V_n and state the discrete system of equations in matrix form.

(4 points)

Exercise 2. (piecewise quadratic finite elements in 1D)

Let $I = [0, 1]$ be the unit interval. We divide I into $n > 2$ subintervals $S_i = [x_i, x_{i+1}]$ with $x_i = i/n$, $i = 0, \dots, n$. Let V_n be the space of continuous functions on I which are piecewise quadratic polynomials on the subintervals, i.e.

$$V_n = \{u \in \mathcal{C}(I) \mid u|_{S_i} \in P_2(S_i) \text{ for } i = 0, \dots, n-1\}\,.$$

a) Determine the dimension of V_n as a vector space. Does a nodal basis exist?

b) Find a basis $\{\phi_0, \dots, \phi_{2n}\}$ of V_n that satisfies the following properties:

- For $0 \le i \le n$, ϕ_i is piecewise linear and satisfies $\phi_i(x_j) = \delta_{ij}$ for $j = 0, \dots, n$.
- For $n + 1 \le i \le 2n$, $\phi_i(x_j) = 0$ for $j = 0, \dots, n$.
  Additionally, $\phi_i(0.5(x_j + x_{j+1})) = \delta_{i-n-1,j}$ for $j = 0, \dots, n-1$.

(4 points)

Programming Exercise 1. (finite differences)

The goal of this programming exercise is to solve the 1D Poisson equation with homogeneous Dirichlet boundary conditions, using finite differences.

a) Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \in \mathbb{R}^{d\times d}.$$

We split $A = D + R$ into its diagonal part $D = 2I$ and remainder R. Implement functions for the matrix-vector multiplication with D^{-1} and R , i.e. routines that take a double precision array y of length d and return the arrays $D^{-1}y$ and Ry, respectively. Make sure that these routines take $\mathcal{O}(d)$ computation time.

b) The Jacobi method solves a linear system of equations

$$Au = b$$

using the fixpoint iteration

$$u^{(k+1)} = D^{-1}(b - Ru^{(k)})\,.$$

Implement the Jacobi method for the given matrix A and general right hand side b and starting vector $x^{(0)}$. As a stopping criterion for the fixpoint iteration, take $\|u^{(k)} - u^{(k-1)}\| < \epsilon$. In other words, write a routine that takes $b \in \mathbb{R}^d$, $x^{(0)} \in \mathbb{R}^d$ and ϵ as input and returns the solution vector $u \in \mathbb{R}^d$.

c) Use this method to solve the following discretized 1D Poisson problem:

$$Au = \frac{1}{(d+1)^2}b\,, \quad b = (1,\ldots,1)^\top \in \mathbb{R}^d$$

Take $\epsilon = 0.0000001$ and test your program for $d = 15$ and $d = 63$. Plot the solution $u \in \mathbb{R}^d$ to the unit interval $I = (0,1)$ via

$$u_i = u(x_i)\,, \quad , x_i = i/(d+1) \text{ for } i = 1,\ldots,d$$

and interpret your results. Which function is being approximated?

(16 points)

The programming exercise should be handed in during the exercise classes (bring your own laptop!) in two weeks, i.e. on 23/24.11.17. All group members need to attend the presentation of your solution to get points.

The student council of mathematics will organize the math party on 23/11 in N8schicht. The presale will be held on Mon 20/11, Tue 21/11 and Wed 22/11 in the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de