Scientific Computing 1

Winter term 2017/18
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Sheet 5

Submission on **Thursday, 23.11.17**.

Exercise 1. (bilinear elements)

Consider the unit square $\Omega = [0,1]^2 \subset \mathbb{R}^2$. We call a continuous function $f\colon \Omega \to \mathbb{R}$ affine bilinear if $f(\cdot, y)$ is affine linear for all $y \in \Omega$ and $f(x, \cdot)$ is affine linear for all $x \in \Omega$.

a) Let $Q(\Omega)$ be the space of affine bilinear functions on Ω. Show that $Q(\Omega)$ has dimension 4, and find a basis which is nodal with respect to the corners of Ω.

Let $n \in \mathbb{N}$. We define $a_i = i/n$ for $i = 1, \dots, n$ and decompose Ω into a union of squares

$$\Omega_{ij} = \{(x, y)^\top \in \Omega \mid a_{i-1} \le x \le a_i,\, a_{j-1} \le y \le a_j\} \subset \Omega$$

for $i, j = 1, \dots, n$.

b) Find the dimension of

$$V = \{f \in \mathcal{C}(\Omega) \mid f|_{\Omega_{ij}} \in Q(\Omega_{ij}) \text{ for } i, j = 1, \dots, n\}$$

and determine whether a nodal basis with respect to the gridpoints $(a_i, a_j)^\top$, $i, j = 0, \dots, n$ exists.

(4 points)

Exercise 2. (change of variable)

We consider the reference triangle element $T_0 = \{(x, y)^\top \in \mathbb{R}^2 \mid x \ge 0, y \ge 0, x + y \le 1\}$ with the three nodes $a_1 = (0,0)^\top$, $a_2 = (1,0)^\top$ and $a_3 = (0,1)^\top$. Furthermore, let ϕ_1, ϕ_2, ϕ_3 be the nodal basis with respect to a_1, a_2, a_3. For an arbitrary nondegenerate triangle $T \subset \mathbb{R}^2$ with corners $b_1, b_2, b_3 \in \mathbb{R}^2$, we consider the affine linear map $J(x, y) = C(x, y)^\top + d$ which maps a_i to b_i for $i = 1, 2, 3$.

a) Determine the matrix $C \in \mathbb{R}^{2\times 2}$ and the vector $d \in \mathbb{R}^2$.

b) We consider new coordinates $(\hat{x}, \hat{y})^\top = J(x, y)$ and define the nodal basis ψ_1, ψ_2, ψ_3 on T via

$$\psi_i(\hat{x}, \hat{y}) = \phi_i(x, y)\,, \quad i = 1, 2, 3\,.$$

Express

$$\int_T \psi_i(\hat{x}, \hat{y})\psi_j(\hat{x}, \hat{y})\,\mathrm{d}(\hat{x}, \hat{y}) \text{ and } \int_T \nabla_{(\hat{x}, \hat{y})}\psi_i(\hat{x}, \hat{y}) \cdot \nabla_{(\hat{x}, \hat{y})}\psi_j(\hat{x}, \hat{y})\,\mathrm{d}(\hat{x}, \hat{y})$$

as integrals of the form

$$\int_{T_0} \cdot \,\mathrm{d}(x, y)\,.$$

(4 points)