Scientific Computing 1

Winter term 2017/18
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Sheet 6

Submission on **Thursday, 30.11.17**.

Exercise 1. (triangular Lagrange element in 2D)

Consider the reference triangle $T_0 = \{(x,y)^\top \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$. For a fixed $n \in \mathbb{N}$, define $z_{i,j} = (i/n, j/n)^\top$ for all $i,j = 0,\ldots,n$ with $i+j \leq n$. Also define $P_n(T_0)$ to be the space of polynomials with order at most n, i.e.

$$P_n(T_0) = \operatorname{span}\{x^i y^j \mid i+j \leq n\}\,.$$

Show that the dimension of $P_n(T_0)$ coincides with the number of nodes $\{z_{i,j}\}$ and find the nodal basis of $P_n(T_0)$ with respect to these nodes.

(4 points)

Exercise 2. (rectangular Lagrange element in 2D)

Consider the reference rectangle $R_0 = [0,1]^2$. For a fixed $n \in \mathbb{N}$, define $z_{i,j} = (i/n, j/n)^\top$ for $i,j = 0,\ldots,n$. Also define $Q_n(R_0)$ to be the space of polynomials of mixed order at most n, i.e.

$$Q_n(T_0) = \operatorname{span}\{x^i y^j \mid i,j \leq n\}\,.$$

Show that the dimension of $Q_n(R_0)$ coincides with the number of nodes $\{z_{i,j}\}$ and find the nodal basis of $Q_n(R_0)$ with respect to these nodes.

(4 points)

Programmieraufgabe 1. (local mass/stiffness matrix)

We consider the reference triangle element $T_0 = \{(x,y)^\top \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y \leq 1\}$ with the three nodes $a_1 = (0,0)^\top$, $a_2 = (1,0)^\top$ and $a_3 = (0,1)^\top$. Furthermore, let ϕ_1, ϕ_2, ϕ_3 be the nodal basis with respect to a_1, a_2, a_3. For an arbitrary nondegenerate triangle $T \subset \mathbb{R}^2$ with corners $b_1, b_2, b_3 \in \mathbb{R}^2$, we consider the affine linear map $J(x,y) = C(x,y)^\top + d$ which maps a_i to b_i for $i = 1,2,3$. With the new coordinates $(\hat{x},\hat{y})^\top = J(x,y)$, the nodal basis ψ_1, ψ_2, ψ_3 on T is given via

$$\psi_i(\hat{x},\hat{y}) = \phi_i(x,y)\,, \quad i = 1,2,3\,.$$

a) Write a routine that takes $b_1, b_2, b_3 \in \mathbb{R}^2$ as input and returns the local mass matrix $M_T \in \mathbb{R}^{3\times 3}$, given by

$$(M_T)_{ij} = \int_T \psi_i(\hat{x},\hat{y})\psi_j(\hat{x},\hat{y})\,\mathrm{d}(\hat{x},\hat{y})$$

for $i, j = 1,2,3$.

b) Write a routine that takes $b_1, b_2, b_3 \in \mathbb{R}^2$ as input and returns the local stiffness matrix $A_T \in \mathbb{R}^{3\times 3}$, given by

$$(A_T)_{ij} = \int_T \nabla_{(\hat{x},\hat{y})}\psi_i(\hat{x},\hat{y}) \cdot \nabla_{(\hat{x},\hat{y})}\psi_j(\hat{x},\hat{y})\,\mathrm{d}(\hat{x},\hat{y})$$

for $i, j = 1,2,3$.

c) Test your implementations for $b_1 = (0.5, 0.5)^\top$, $b_2 = (0,0)^\top$, $b_3 = (1,0)^\top$.

(16 points)

The programming exercise should be handed in during the exercise classes (bring your own laptop!) in two weeks, i.e. on 7.12/8.12.17. All group members need to attend the presentation of your solution to get points.