Numerical Algorithms

Winter semester 2018/2019
Prof. Dr. Marc Alexander Schweitzer
Denis Düsseldorf

universitätbonn

Exercise sheet 1. Submission on **Tuesday, 2018-10-23, before the lecture**.

Exercise 3. (Gaussian quadrature)

We recall some properties of one dimensional Gaussian quadrature with respect to some weight $\omega : [a, b] \to (0, \infty)$. Assume that $(x_i, w_i) \in [a, b] \times (0, \infty)$, $0 < i \leq n$ are the nodes and weights of an interpolatoric quadrature rule, i.e.

$$\int_{[a,b]} \omega f \approx Q(f) := \sum_{i=0}^{n} w_i f(x_i) \; .$$

Consider the node polynomial

$$p_n(x) := \prod_{i=0}^{n} (x - x_i) \; .$$

Show that

a) Q exactly integrates at most all polynomials of degree $2n + 1$.

b) Q exactly integrates all polynomials of degree $2n+1$ if and only if for all polynomials q of degree $\leq n$

$$\int p_n q \omega = 0 \; .$$

c) If Q is exact for degree $2n + 1$, then it holds that $w_i > 0$.

(3 Points)

Exercise 4. (Green's identities)

Let $\Omega \subset \mathbb{R}^2$ be a simply connected smooth domain with smooth boundary $\partial\Omega$. Let $u, A \in \mathbb{C}^2(\Omega)$ and $v \in \mathbb{C}^1(\Omega)$.

a) Show that

$$\int_\Omega \Delta u v \mathrm{d}x = \int_{\partial\Omega} v \nabla u \cdot n \mathrm{d}S - \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x \; .$$

b) Show that

$$\int_\Omega \nabla \cdot (A \nabla u) \, v \mathrm{d}x = \int_{\partial\Omega} v A \nabla u \cdot n \mathrm{d}S - \int_\Omega A \nabla u \cdot \nabla v \mathrm{d}x \; .$$

(3 Points)

Exercise 5. (Weak derivative)

Consider a piecewise smooth function

$$u(x) = \begin{cases} v(x) \; , & x \in (0, a] \\ w(x) \; , & x \in (a, b) \end{cases} \qquad\qquad v, w \in C^1([0, 1])$$

on the interval $(0, 1) \ni a$.
Show that $u \in H^1$ if and only if $v(a) = w(a)$.

(3 Points)

Exercise 6. (Weak form of a mixed boundary value problem)

Let Ω be a bounded domain with sufficiently smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Consider the PDE

$$-\nabla \cdot (\kappa \nabla u) = f \;, \qquad x \in \Omega$$

with mixed boundary values

$$\begin{aligned} u &= g \;, & x &\in \Gamma_1 \;, \\ \kappa \frac{\partial u}{\partial \nu} &= h \;, & x &\in \Gamma_2 \;. \end{aligned}$$

a) Determine a suitable space of test functions for the weak form of this boundary value problem.

b) Derive the weak form of this boundary value problem.

(3 Points)

Programming exercise 2. (Simple mesh data structures)

We will extend the programming done in programming exercise 1.
Furthermore, we recommend you to get to know `scipy.sparse`, in particular `scipy.sparse.linalg` and their documentation for future exercises.

a) Assume a set of nodal values $u_i = u(x_i, y_i)$ of some function u at the mesh vertices (x_i, y_i) is given. Write a function for a wireframe plot of u based on the functions from programming exercise 1 and vertices $(x_i, y_i, u_i) \in \mathbb{R}^3$. Hint: `mpl_toolkits.mplot3d`

b) With the functions from programming exercise 1 create and plot a mesh of $[-1, 1]^2$.

c) With a), plot $u(x, y) = xy$ on $[-1, 1]^2$.

(4 Points)

Send to `duesseldorf@ins.uni-bonn.de`