Numerical Algorithms

Winter semester 2018/2019
Prof. Dr. Marc Alexander Schweitzer
Denis Duesseldorf

universitätbonn

Exercise sheet 6.

Submission on **Tuesday, 2018-11-27, before lecture.**

Exercise 18. (Richardson iteration)

In the lecture it was already recalled that every consistent, linear iteration Φ with regular N, W

$$Ax = b\ , \qquad\qquad x^{m+1} = x^m - W^{-1}(Ax^m - b)\ .$$

is equivalent to the Richardson iteration with damping $\theta = 1$ of the preconditioned system

$$\underbrace{W^{-1}A}_{=:B}\,x = \underbrace{W^{-1}b}_{=:c}\ , \qquad\qquad x^{m+1} = x^m - \theta(Bx^m - c)\ .$$

which allows us to use the convergence results for the Richardson iteration for many other linear iterations.

Let $M_R^\theta = I - \theta B$ be the iteration matrix of the Richardson iteration Φ_R. Assuming that B has only positive eigenvalues, $\sigma(B) \subset (0, \infty)$, show the following.

a) $\rho(M_R^\theta) = \max\{|1 - \theta\lambda_{min}(B)|, |1 - \theta\lambda_{max}(B)|\}$

b) Φ_R converges if and only if $0 < \theta < \frac{2}{\lambda_{max}(B)}$.

c) The optimal convergence rate, i.e. $\rho(M_R^\theta) = \min_{\tilde\theta} \rho(M_R^{\tilde\theta})$ is obtained for

$$\theta = \frac{2}{\lambda_{max}(B) + \lambda_{min}(B)}\ , \qquad\qquad \rho(M_R^\theta) = \frac{\lambda_{max}(B) - \lambda_{min}(B)}{\lambda_{max}(B) + \lambda_{min}(B)}\ .$$

d) If $B > 0$ then Φ_R converges if and only if $0 < \theta < \frac{2}{\|B\|_2}$.

e) Using the condition $\kappa(B) = \lambda_{max}(B)/\lambda_{min}(B)$ yields optimal

$$\theta = \frac{2\|B^{-1}\|_2}{\kappa(B) + 1}\ , \qquad\qquad \rho(M_R^\theta) = \frac{\kappa(B) - 1}{\kappa(B) + 1}$$

f) How do a)–e) relate to the iteration Φ with A, W? How can these results also be used for convergence of the undamped case $\theta = 1$.

(3 Points)

Exercise 19. (Some condition estimates)

In lecture and the previous exercise we have seen that convergence criteria and rates boil down to spectral radius $\rho(M)$ or condition number $\kappa(W^{-1}A)$. In this exercise we work with some equivalencies and estimates for the condition number. Let $A > 0, B > 0$. Show the following.

a) The following, with some $\alpha, \beta > 0$, are equivalent

(a) $\alpha A \leq ABA \leq \beta A$

(b) $\beta^{-1}A \leq B^{-1} \leq \alpha^{-1}A$

(c) $\alpha B \leq BAB \leq \beta B$

(d) $\beta^{-1}B \leq A^{-1} \leq \alpha^{-1}B$

$$\alpha A \leq ABA \leq \beta A\ , \qquad \alpha B \leq BAB \leq \beta B\ ,$$
$$\beta^{-1}A \leq B^{-1} \leq \alpha^{-1}A\ , \qquad \beta^{-1}B \leq A^{-1} \leq \alpha^{-1}B\ .$$

b) If any of the equivalent inequalities from a) hold then $\kappa(BA) \leq \frac{\beta}{\alpha}$.

Hint: If $A > 0$ then we not only have $\sigma(A) \subset (0, \infty)$, but also that the maximal and minimal eigenvalues can directly be given by Raleigh-quotients/numerical values of A, i.e.

$$\lambda_{min}(A) = \min_{v \neq 0} \frac{< Av, v >}{< v, v >}\ , \qquad \lambda_{max}(A) = \max_{v \neq 0} \frac{< Av, v >}{< v, v >}\ .$$

(3 Points)

Exercise 20. (PSC)

Consider a (sub)space decomposition $V = \sum_{i=1}^{J} V_i$ of some inner product space $(V, (\cdot, \cdot))$ with operators

$$\begin{aligned}
Q_i &: V \to V_i\ , & (Q_i u, v_i) &= (u, v_i)\ , & u \in V, v_i \in V_i\ ,\\
P_i &: V \to V_i\ , & (P_i u, v_i)_A &= (AP_i u, v_i) = (Au, v_i) = (u, v_i)_A\ , & u \in V, v_i \in V_i\ ,\\
A_i &: V_i \to V_i\ , & (A_i u_i, v_i) &= (Au_i, v_i)\ , & u_i, v_i \in V_i\ ,\\
R_i &: V_i \to V_i\ , & R_i &\approx A_i^{-1}\ .
\end{aligned}$$

with $A > 0, R_i > 0$.

a) Show that $N_{PSC} = \sum_{i=1}^{J} R_i Q_i > 0$.

b) Give $V, V_i, (\cdot, \cdot), R_i$ such that PSC yields the additive Schwarz method for some choice of $\Lambda, p_\lambda, \cdots$.

c) In the lecture it was established that $\kappa(N_{PSC}A) = \kappa(BA) < K_0 K_1$ with K_0, K_1 from preconditions 1) and 2). Show that $K_1 < 2$ is sufficient for convergence.

(3 Points)

Exercise 21. (PCG)

In previous lectures we have already seen the preconditioned CG-method based on a right preconditioning

$$Ax = b \Rightarrow ACy = Cb, x = Cy$$

and using the inner product $< x, Cy >$.
Derive the CG iteration steps for the left preconditioned system

$$W^{-1}Ax = W^{-1}b$$

in the style of the iterations seen in the lecture using the auxiliary expressions

$$r^k = b - Ax^k \ , \qquad q^k = W^{-1}r^k \ , \qquad \rho^k = \left\langle q^k, r^k \right\rangle \ .$$

Reminder: The usual CG-method for $Ax = b$ has initialization

$$r^0 = b - Ax^0 \ , \qquad p^0 = r^0$$

and iteration steps

$$\alpha^k = \frac{\langle r^k, r^k \rangle}{\langle Ap^k, p^k \rangle} \ , \qquad x^{k+1} = x^k + \alpha^k p^k \ , \qquad r^{k+1} = r^k - \alpha^k Ap^k \ ,$$

$$\beta^k = \frac{\langle r^{k+1}, r^{k+1} \rangle}{\langle r^k, r^k \rangle} \ , \qquad p^{k+1} = r^{k+1} + \beta^k p^k \ .$$

(3 Points)

Just a sidenote: Instead of considering a left- and right-preconditioned system, it is also possible to derive a PCG formulation based on the system $W^{-\frac{1}{2}}AW^{-\frac{1}{2}}x = W^{-\frac{1}{2}}b$. The matrix $W^{-\frac{1}{2}}AW^{-\frac{1}{2}}$ is self-adjoint in the standard scalar product.

Programming exercise 7. (PCG with Schwarz preconditioners)

With programming exercises 1-6 we now have everything in place for a comparision of some iterative schemes for the solution of the linear system arising from the FEM on our decomposed mesh. Consider the domain

$$\Omega_L := (-1,1)^2 \setminus (0,1)^2 \ , \quad \Gamma_D := \partial[-1,1]^2 \cap \partial\Omega_L \ , \quad \Gamma_N := \partial\Omega_L \setminus \Gamma_D = [0,1]^2 \cap \partial\Omega_L$$

underlying the PDE

$$\begin{aligned} -\Delta u(x,y) &= -6 \ , && x \in \Omega_L \ , \\ u(x,y) &= 1 + x^2 + 2y^2 \ , && x \in \Omega_D \ . \end{aligned}$$

and the (non)overlapping domain decompositions from the previous exercise.
In the remaining section we continue to work with the restrictions, prolongations and projections from programming exercise 6

- Implement the PCG method from exercise 21 including preemptive abortion after having reaching some criterion $\|r^k\| < \epsilon$.
- Implement a function for single steps of the damped additive Schwarz iteration Φ^{θ}_{add} with θ chosen to fulfill the convergence criterium for Φ^{θ}_{add}.
- Using exercise 10, implement an efficient function for single steps of the multiplicative Schwarz iteration Φ_{mult}.
- Compare the condition numbers $\kappa(K)$,$\kappa(D^{-1}K)$,$\kappa((W^{\theta}_{add})^{-1}K)$ and $\kappa(W^{-1}_{mult}K)$.
- Compare the errors, iteration numbers and computation times for PCG with $W = I$, $W = D$, $W = W^{\theta}_{add}$ and $W = W_{mult}$, the latter two for both the overlapping and non-overlapping case. Plot error against iteration number and computation time.

Remark: Make sure to modify or post-process the restrictions and prolongations from previous exercises to only cover the degrees of freedom that remain after having introduced Dirichlet boundary conditions, i.e. make sure that the resulting matrices have the right size and indices.

(4 Points)

Send to `duesseld@ins.uni-bonn.de`