Numerical Algorithms

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Prof. Dr. Marc Alexander Schweitzer
Denis Duesseldorf

universitätbonn

Exercise sheet 8.

Submission on **Tuesday, 2016-12-11, after lecture**.

Exercise 26. (Barycentric Triangle coordinates)

Consider the triangle $T = [a_0, a_1, a_2] := \text{conv}(a_0, a_1, a_2) \subset \mathbb{R}^2$ given as the convex hull of vertices $a_0, a_1, a_2 \in \mathbb{R}^2$, $a_i = (x_i, y_i)$, see exercise 9. The barycentric coordinate $\lambda_i(x)$ with respect to vertex a_i is defined as the unique linear polynomial

$$\lambda_i \in P^1(T) \qquad \text{such that} \qquad \lambda_i(a_j) = \delta_{i,j} \qquad \forall j \in \{0, 1, 2\} \; .$$

It follows immediately that $\sum \lambda_i = 1$.
Let $a = (x, y) \in T$ and show the following.

a) The $\lambda_i(a) = \lambda_i(x, y)$ are the solutions of

$$\begin{pmatrix} 1 & 1 & 1 \\ x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix} \begin{pmatrix} \lambda_0(x,y) \\ \lambda_1(x,y) \\ \lambda_2(x,y) \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \; .$$

The solution of this system exists if T is nondegenerate. Compute the explicit solution of this system.

b) The triangle T can be expressed as

$$T = \{a \in \mathbb{R}^2 : \forall i : 0 \le \lambda_i \le 1\} \; .$$

c) The barycentric coordinates are so called natural/area coordinates, i.e. with $|T| = |[a_0, a_1, a_2]|$ denoting the area of the triangle T and $a \in T$ we have

$$\lambda_i(a) = \frac{|[a, a_{i+1 \mod 3}, a_{i+2 \mod 3}]|}{|T|} \; .$$

d) Let F_T be a bijective affine transformation (as in exercise 9). Let $\hat{\lambda}_i$ be the barycentric coordinates of $\hat{T} = (F_T(a_0), F_T(a_1), F_T(a_2))$. Then for any $a \in \mathbb{R}^2$

$$\hat{\lambda}_i(F_T(a)) = \lambda_i(a) \; .$$

e) For the reference triangle

$$T = \{(x, y) : 0 \le x, y \le 1, 0 \le x + y \le 1\} = [(0,0), (1,0), (0,1)]$$

we have

$$\lambda_0(a) = 1 - x - y \qquad \lambda_1(a) = x \qquad \lambda_2(a) = y \; .$$

Hint: Matrices from Exercise 9 and Cramer's rule.

(3 Points)

Exercise 27. (Recurrence relations for Legendre-type polynomials)

For the construction of higher order shape functions we often require rescaled versions of the regular Legendre polynomials L_n and integrated Legendre polynomials $\hat{L}_n$. Show that the following statements hold true.

a) The Legendre polynomials satisfy

$$\begin{aligned} L_0(x) &= 1 \ , \\ L_1(x) &= x \ , \\ (n+1)L_{n+1}(x) &= (2n+1)xL_n(x) - nL_{n-1}(x) \text{ for } n \geq 1 \ . \end{aligned}$$

b) The integrated Legendre polynomials satisfy

$$\begin{aligned} \hat{L}_1(x) &= x + 1 \ , \\ \hat{L}_2(x) &= \frac{1}{2}(x^2 - 1) \ , \\ (n+1)\hat{L}_{n+1}(x) &= (2n-1)x\hat{L}_n(x) - (n-2)\hat{L}_{n-1}(x) \text{ for } n \geq 2 \ . \end{aligned}$$

c) The scaled Legendre polynomials

$$L_n^S(x,t) := t^n L_n\left(\frac{x}{t}\right) \qquad \text{for } x \in [-t,t], t \in (0,1]$$

are polynomials and satisfy

$$\begin{aligned} L_0^S(x,t) &= 1 \ , \\ L_1^S(x,t) &= x \ , \\ (n+1)L_{n+1}^S(x,t) &= (2n+1)xL_n^S(x,t) - nt^2 L_{n-1}^S(x,t) \text{ for } n \geq 1 \ . \end{aligned}$$

d) The scaled integrated Legendre polynomials

$$\hat{L}_n^S(x,t) := t^n \hat{L}_n\left(\frac{x}{t}\right) \text{ for } n \geq 2, x \qquad \in [-t,t], t \in (0,1] \ .$$

satisfy

$$\hat{L}_n^S(x,t) = \int_{-t}^{x} L_{n-1}^S(s,t)\mathrm{d}s$$

and

$$\begin{aligned} \hat{L}_1^S(x,t) &= x + t \ , \\ \hat{L}_2^S(x,t) &= \frac{1}{2}(x^2 - t^2) \ , \\ (n+1)\hat{L}_{n+1}^S(x,t) &= (2n-1)x\hat{L}_n^S(x,t) - (n-2)t^2\hat{L}_{n-1}^S(x,t) \text{ for } n \geq 2 \ . \end{aligned}$$

e) The derivatives L_n', $\hat{L}'$, $(L_n^S)'$, $(\hat{L}_n^S)'$ also satisfy 3 term recurrences.

(4 Points)

Exercise 28. (A higher order triangular element)

For finite elements $(T = [a_0, a_1, a_2], P = \{\varphi_i\}, \Sigma)$ with shapes functions being determined by interpolation at regular nodes, it is difficult to guarantee continuity across elements if the polynomial degree is not constant across triangles. A standard approach to alleviate this problem in p-FEM and hp-FEM are V-E-C shape functions. These allow the polynomial degree (of P) to be chosen independently on edges and vertices. The local space is now decomposed into $P = V + E + C$.

V are vertex-based shape functions

$$V \ni \varphi_i^V(x) := \lambda_i(x) \qquad \text{for } i = 0, 1, 2 \,.$$

E are edge-based shape functions which have polynomial traces on one edge and vanish on the other two edges. A standard choice are scaled Integrated Legendre polynomials of order p_e for all edges $e = [a_k, a_l]$.

$$E \ni \varphi_i^e := \hat{L}_{i+2}^S(\lambda_l - \lambda_k, \lambda_l + \lambda_k) \qquad \text{for } 0 \le i \le p_e - 2 \,.$$

To avoid the so called orientation problem, it can be required for a_k, a_l in the edge e to be sorted according to their global index.

C are cell-based shape functions vanishing on the boundary of T

$$C \ni \varphi_{i,j}^T := \hat{L}_{i+2}^S(\lambda_1 - \lambda_0, \lambda_1 + \lambda_0)\lambda_2 L_j(2\lambda_2 - 1) \,. \quad \text{for } 0 \le i + j \le p_T - 3, i, j \ge 1 \,.$$

Due to this decomposition of P we now have variable polynomial order $(p_{e_0}, p_{e_1}, p_{e_2}, p_T)$. Similarly, in three dimensions, the local space P for a p-FEM or hp-FEM can be split into vertex-based, edge-based, face-based and cell-based approximation spaces.

Show the following.

a) For all $\varphi_i^e \in E$ and $\forall j : \varphi_i^e(a_j) = 0$.

b) Let e_l be the edges of T and, without loss of generality, $e_k = [a_0, a_1]$

$$\varphi_i^{e_k}|_{e_l} = \begin{cases} \hat{L}_{i+2}\left(\frac{\lambda_1 - \lambda_0}{\lambda_1 + \lambda_0}\right) & k = l \\ 0 & k \neq l \end{cases}$$

c) The $\varphi_{i,j}^T$ are so called bubble functions, i.e. $\varphi_{i,j}^T \in P_0^{p_T}(T)$. That is, they are polynomials of maximal total degree p_T with

$$\varphi_{i,j}^T|_{e_l} = 0 \qquad \forall e_l \,.$$

d) Using the Duffy transform D from exercise 25 yields

$$(\varphi_{i,j}^T \circ D)(\xi, \eta) = \hat{L}_{i+2}(\xi) \left(\frac{1 - \eta}{2}\right)^{i+2} \frac{\eta + 1}{2} L_j(\eta)$$

e) The local V-E-C shape functions are H^1 conforming and linearly independent. For uniform order $p = p_{e_l} = P_C$ they form a basis of $P^p(T)$.

   Hint: Show linear independence and use a counting argument.

f) For the reference triangle compute the 6 shape functions $\varphi_i \in P$ for $p = 2$.

(5 Points)

Programming exercise 9. (Legendre polynomials)

For later use we will now put the recursion formulas for the scaled and or integrated Legendre polynomials into code. This code can be used for the implementation of the higher order shape functions of the previous exercise.

a) Write a function for the efficient point evaluation of L_n^S, $(L_n^S)'$ and plot $L_n^S(\cdot, 1)$ for $n \leq 5$ over $[-1, 1]$.

b) Write a function for the efficient point evaluation of $\hat{L}_n^S$, $(\hat{L}_n^S)'$ and plot $\hat{L}_n^S(\cdot, 1)$ for $2 \leq n \leq 6$ over $[-1, 1]$.

c) Using exercise 26 and a),b), implement the point evaluation of the V-E-C shape functions from the previous exercise for arbitrary triangles.

d) Consider the triangle $T = [(-1, -1), (1, -1), (0, 1)] = [a_0, a_1, a_2]$. Plot the shape functions φ_i^V, φ_i^e for $e = [a_1, a_2]$, and φ_i^T for $p = p_e = p_T = 4$.

   Hint: Red-refine the triangle and use programming exercise 2a). Alternatively, use `numpy.meshgrid`, the Duffy transformation and `pyplot.plot_wireframe` .

(4 Points)

Send to `duesseldorf@ins.uni-bonn.de`