Numerical Algorithms

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Exercise sheet 9.

Submission on **Monday, 2018-12-18, after lecture**.

Exercise 29. (Cubic Hermite splines)

Consider a partioning of $[a, b]$ into N intervals using nodes

$$a = x_0 < x_1 < \cdots < x_N = b \, .$$

Assume that data $f_i, i = 0, \ldots, N$ are given. For a) and b) further assume that $f_i', i = 0, \ldots, N$ are given.

a) Let $a = 0$,$b = 1$ and $N = 1$. Show that

$$\begin{aligned}
\alpha_1(t) &:= 2t^3 - 3t^2 + 1\\
\alpha_2(t) &:= -2t^3 + 3t^2\\
\alpha_3(t) &:= t^3 - 2t^2 + t\\
\alpha_4(t) &:= t^3 - t^2\\
\Rightarrow p(t) &:= f_0\alpha_1(t) + f_1\alpha_2(t) + f_0'\alpha_3(t) + f_1'\alpha_4(t)
\end{aligned}$$

yields a solution $p \in \mathcal{P}_3$ of the Hermite interpolation problem

$$p(0) = f_0 \, , \qquad p(1) = f_1 \, , \qquad p'(0) = f_0' \, , \qquad p'(1) = f_1' \, .$$

b) For arbitrary x_i we define $p_i(x)$ on each $[x_{i-1}, x_i], i = 1, \ldots N$ via

$$h_i := x_i - x_{i-1} \, , \qquad t_i(x) := (x - x_{i-1})/h_i \, , \qquad \alpha_{k,i} := \alpha_k(t_i(x))$$

with $i = 1, \ldots N$, $k = 1, \ldots 4$.

Use $\alpha_{k,i}$ to construct $p_i \in \mathcal{P}_3$ such that

$$p_i(x_{i-1}) = f_{i-1} \, , \qquad p_i(x_i) = f_i \, , \qquad p_i'(x_{i-1}) = f_{i-1}' \qquad p_i'(x_i) = f_i' \, .$$

c) With a) and b) we can now construct the Hermite Spline interpolant. With given data f_i and f_0', f_N', the missing values $f_i', 0 < i < N$ are chosen such that the resulting piecewise polynomial

$$p(x) = p_i(x) \qquad\qquad x \in [x_{i-1}, x_i]$$

is twice continously differentiable.

Show that $p \in C^1$ even without a proper choice of f_i'. Compute $p_i^{(2)}$ in x_{i-1}, x_i and establish a system of linear equations for f_i' to yield $p \in C^2$.

d) How does the linear system from c) change when requiring natural boundary conditions

$$p_1''(a) = p_N''(b) = 0$$

instead of the Hermite boundary conditions $p_1'(a) = f_0', p_N'(b) = f_N'$?

(4 Points)

Exercise 30. (B-Splines)

The $0 < n$ B-Spline basis functions $N_{i,p}, i = 1, \dots, n$ of polynomial degree p corresponding to a so called knot vector

$$\Xi = (\xi_1, \xi_2, \dots, \xi_{n+p+1}) \,, \qquad\qquad \xi_1 \le \xi_2 \le \cdots \le \xi_{n+p+1}$$

are typically given via the recursion

$$N_{i,0}(\xi) = \begin{cases} 1 & \xi_i \le \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases} \,,$$
$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \,.$$

with the convention of $\frac{0}{0} = 0$ in case where $\xi_i = \xi_{i+p}, N_{i,p-1} = 0$ or $\xi_{i+p+1} = \xi_{i+1}, N_{i+1,p-1} = 0$, see also c).
Using these functions and so called control points $B_i \in \mathbb{R}^d, i = 1, \dots, n$ the piecewise-polynomial B-spline curve is given as

$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) B_i \,.$$

Show the following properties.

a) If the knots are distinct, $\forall i : \xi_i < \xi_{i+1}$, then the B-spline basis admits the explicit representation

$$N_{i,p} = (\xi_{i+p+1} - \xi_i) \sum_{j=i}^{i+p+1} \left(\prod_{k=i,k\neq j}^{i+p+1} (\xi_j - \xi_k) \right)^{-1} (\xi_j - \xi)_+^p$$

where

$$(\xi_j - \xi)_+^p = \begin{cases} (\xi_j - \xi)^p & \xi_j > \xi \\ 0 & \text{otherwise} \end{cases} \,.$$

Hint: Show that the functions defined this way satisfy the recursion given for the B-spline basis.

b) $N_{i,p}$ depends only on the local knots

$$\{\xi_i, \dots, \xi_{i+p+1}\}$$

c) The local support property

$$N_{i,p}(\xi) = 0 \,, \qquad\qquad \xi \notin [\xi_i, \xi_{i+p+1})$$

holds.

In particular

$$\xi_i = \xi_{i+p+1} \Rightarrow N_{i,p} \equiv 0$$

and, if $\xi \in [\xi_j, \xi_{j+1})$,

$$(i < j - p) \vee j < i \qquad \Rightarrow \qquad N_{i,p}(\xi) = 0 \,.$$

d) The basis is positive inside its support

$$N_{i,p}(\xi) > 0 \,, \qquad\qquad \xi \in (\xi_i, \xi_{i+p+1}) \,.$$

e) The B-spline basis functions are piecewise polynomials

$$N_{i,p}(\xi) = \sum_{j=i}^{i+p} N_{i,p}^{j}(\xi) N_{j,0}(\xi)$$

with $N_{i,p}^{j}$ being polynomials of degree at most p.

f) If $z = \xi_{i+1} = \ldots = \xi_{i+p} < \xi_{i+p+1}$ then

$$N_{i,p}(z) = 1 \,, \qquad \text{and} \qquad N_{j,p}(z) = 0 \qquad \forall j \neq i \,.$$

g) If z occurs k times among $\xi_i, \ldots, \xi_{i+p+1}$ then

$$N_{i,p} \in C^{p-k} \,.$$

In particular $N_{i,p} \in C^{p-1}$ if ξ_i are distinct.

h) The B-spline basis forms a partition of unity on $[\xi_{p+1}, \xi_{n+1})$

$$\sum_{i=1}^{n} N_{i,p}(\xi) = 1 \,, \qquad \xi \in [\xi_{p+1}, \xi_{n+1}] \,.$$

i) If $\xi \in [\xi_j, \xi_{j+1})$ with $p + 1 \leq j \leq n$ then

$$C(\xi) = \sum_{i=j-p}^{j} N_{i,p}(\xi) B_i$$

j) If $z = \xi_{i+1} = \ldots = \xi_{i+p} < \xi_{i+p+1}$ with $1 \leq i \leq n$ then

$$C(z) = B_i \,.$$

k) If z occurs k times in Ξ then C has continuous derivatives to order $p - k$.

(8 Points)

Programming exercise 10. (Splinoraptor)

A recent excursion into the Siebengebirge brought to light remains of what is believed to be a petrified fossil. For the reconstruction of the shape of the original animal the 20×10 unit wide excavation site is covered with a grid. A logical reordering of the coordinates of the fossil remains yields the following $c_i = (x_i, y_i)$.

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
x_i	13	11.5	10	11	9	6	0	6	10.5	13	15	15	16	19
y_i	0	0	2	4	5.5	6	5	7	7	7.5	7	9	10	9

i	14	15	16	17	18	19	20	21	22	23	24	25	26
x_i	17	16.5	16	15	15	15.5	14.5	15	14	13	12	11	13
y_i	8.5	8	7	6	4.5	3.5	2.5	3.5	4.5	4.5	2.5	1.5	0

Help reconstruct possible shapes of the fossil.

a) Compute and plot the Hermite spline interpolants with natural boundary conditions, see 29e), for each of the components, i.e. find

$$p(t_i) = x_i \,, \qquad p'(t_0) = p'(t_{26}) = 0 \,; \qquad q(t_i) = y_i \,, \qquad q'(t_0) = q'(t_{26}) = 0$$

and plot the curve

$$\{(p(\xi), q(\xi)) : \xi \in [t_0, t_{26}]\}$$

for the two parametrizations

$$t_i = i \,; \qquad t_0 = 0 \,, \qquad t_{i+1} = t_i + \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2} \,.$$

The resulting curve will C^2 smooth.

Hint: For an efficient implementation recall the Thomas-algorithm for tridiagonal matrices.

b) Plot the B-spline basis of order $p = 3$ for the knot vector

$$(0, 0, 0, 0, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 8, 9, 10, 11, 11, 11, 12, 13, 13, 14, 15, 16, 16, 16, 16)$$

c) Using b) plot the cubic B-spline curve through $B_i = c_{i-1}$.

Note how the choice of knots, in particular the multiplicities, affect the resulting contour.

(4 Points)

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