Numerical Algorithms

Winter semester 2018/2019
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Exercise sheet 11. Submission on **2019-01-22, after lecture.** **Submission is not obligatory. However, you can gain additional points if you do hand in.**.

Exercise 35. (Bézier curves)

We introduce a special case of B-spline basis functions commonly used for graphics applications, the so called Bézier curves. To emphasize the dependence of the B-spline functions $N_{i,p}$ on the knots $\xi_i, \dots, \xi_{i+p+1}$ we introduce the notation

$$N_{i,p}(\xi) = N(\xi|\xi_i, \dots, \xi_{i+p+1}) \, .$$

Based on the interval $[0, 1] \ni \xi$, $d > 0$ and the $d + 1$ control points $c_i, i = 0, \dots, d$ we define the degree d Bézier curve through the points c_i via the recursion

$$\begin{aligned} p_{i,0}(\xi) &= p(\xi|c_i) := c_i \\ p_{i,r}(\xi) &= p(\xi|c_{i-r}, \dots, c_i) \\ &:= (1-\xi)p(\xi|c_{i-r}, \dots, c_{i-1}) + \xi p(\xi|c_{i-r+1}, \dots, c_i) \\ &= (1-\xi)p_{i-1,r-1}(\xi) + \xi p_{i,r-1}(\xi) \end{aligned} \tag{1}$$

for $i = r, \dots, d$ and $r = 1, 2, \dots, d$.
A composite Bézier curve on $[0, N]$ is then obtained by taking control points $c_i^j, \dots, c_d^j$ for $j = 1, \dots, N$ with

$$\begin{aligned} c_d^{j-1} &= c_0^j \, , & j &\in \{2, \dots, N\} \\ p(\xi) &= p(\xi - j|c_0^j, \dots, c_d^j) \, , & \xi &\in [j-1, j] \end{aligned}$$

Show the following.

a) It holds that

$$p_{d,d}(\xi) = \sum_{i=0}^{d} B_{i,d}(\xi)c_i \qquad \text{where} \qquad B_{i,d}(\xi) := \binom{d}{i}\xi^i(1-\xi)^{d-i} \, .$$

The $B_{i,d}$ are referred to as *Bernstein polynomials*.

b) The Bernstein polynomials are special cases of B-splines, except for the cutoff outside $[0, 1)$, namely

$$B_{i,d}(\xi)N(\xi|0,1) = N(\xi|\underbrace{0,\dots,0}_{d+1-i},\underbrace{1,\dots,1}_{i+1})$$

for $i = 0, \dots, d$.

They fulfill the recursion

$$B_{i,d}(\xi) = \xi B_{i-1,d-1}(\xi) + (1-\xi)B_{i,d-1} \, .$$

c) The curve $p_{d,d}$ interpolates the first and last control points c_0 and c_d.

$$p_{d,d}(0) = c_0 \ , \qquad\qquad p_{d,d}(1) = c_d \ .$$

d) The tangents at the ends points c_0, c_d point in the direction from c_0 to c_1 and c_{d-1} to c_d.

$$p'_{d,d}(0) = d(c_1 - c_0) \ , \qquad\qquad p'_{d,d}(1) = d(c_d - c_{d-1})$$

e) A composite Bézier curve of degree $d > 0$ will be C^1 continuous if and only if for all $1 < j \leq N$

$$c_d^{j-1} - c_{d-1}^{j-1} = c_1^j - c_0^j$$

(4 Points)

Exercise 36. (NURBS)

A commonly found extension of B-Splines and B-Spline curves that allows for the perfect representation of all conic section, e.g. circles, are NURBS (Non-uniform rational B-spline). In addition to knots $\xi_1, \dots, \xi_{i+p+1}$ and control points $B_1, \dots, B_n$ we now have non-negative weights $w_1, \dots, w_n$.
With rational basis functions

$$R_{i,p}(\xi) := \frac{N_{i,p}(\xi) w_i}{\sum_{j=1}^n N_{j,p}(\xi) w_j}$$

the NURBS curve to control points B_i is given as

$$D(\xi) := \sum_{i=1}^N R_{i,p}(\xi) B_i$$

Most properties carry over from B-splines, if necessary replacing polynomials with rational functions. Among the additional properties we have the following.

a) Show that

$$R_{i,p}(\xi; w_i \to 0) = 0 \ , \qquad R_{i,p}(\xi; w_i \to \infty) = 1 \ , \qquad R_{i,p}(\xi; w_j \to \infty, j \neq i) = 0 \ .$$

Moreover, if $w_i \to \infty$ then

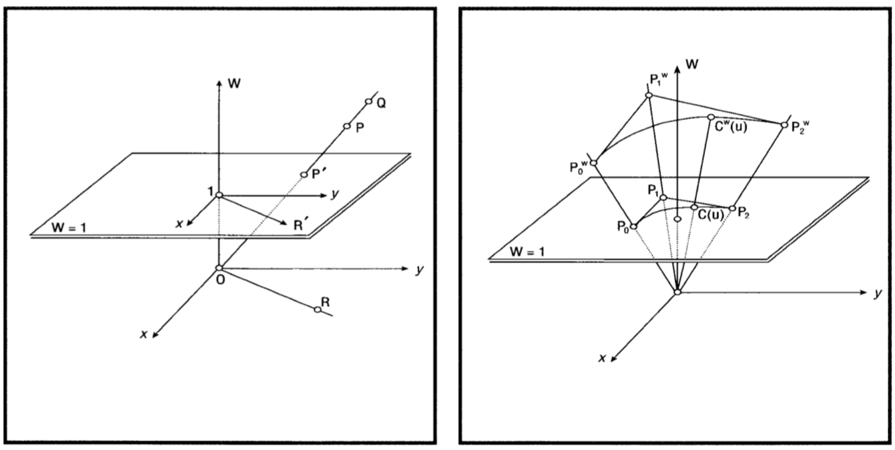

Figure 1: (left) Euclidean model of the projective plane. (b) Geometric construction of NURBS curves.

b) NURBS with $B_i = (b_{i,j})_{j=1}^d \in \mathbb{R}^d$ can be interpreted as B-Splines with $\tilde{B}_i = (w_i b_{i,1}, \ldots, w_i b_{i,d}, , w_i) \in \mathbb{R}^{d+1}$ projected to the plane $\mathbb{R}^d \times \{1\}$ with a projective transformation through $O = (0,0,0)$. For ease of notation and visualization we restrict ourselves to $B_i = (x_i, y_i) \in \mathbb{R}^2, \tilde{B}_i = (w_i x_i, w_i y_i, w_i) \in \mathbb{R}^3$, see also Figure 1. Denote the coordinate axes of $\mathbb{R}^3$ with X, Y and W and let x, y be the axes of another coordinate system centered at $(X, Y, W) = (0,0,1) \in \mathbb{R}^3$ with x parallel to X and Y parallel to Y. We call the plane spanned by the x, y axis the projective plane. Every point P' in the projective plane determines a line OP', and every line passing through O not lying in the X, Y plane determines a point in the projective plane. This line can be defined by any point P or Q and any of its coordinates $(XP, YP, WP), (XQ, YQ, WQ) \in \mathbb{R}^3$, so called homogeneous coordinates of P', define the same line through P'. The perspective/projective mapping of $\mathbb{R}^3 \setminus \{W = 0\}$ into the projective plane is given by

$$(x, y) = \varphi(X, Y, W) = \frac{1}{W}(X, Y) .$$

Show that, with the regular, nonrational, B-spline curve

$$\tilde{C}(\xi) = \sum_{i=1}^{N} N_{i,p}(\xi) \tilde{B}_i$$

yields, using the perspective mapping φ the NURBS curve in $\mathbb{R}^2$ (not to be confused with the projective plane $(x, y, 1)$)

$$D(\xi) = \varphi(\tilde{C}(\xi)) .$$

c) The circle can be given as the quadratic NURBS curve with

$$\begin{aligned} \Xi &:= \{0,0,0,\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},1,1,1\} , \\ B_i &:= \{(1,0),(1,1),(-1,1),(-1,0),(-1,-1),(1,-1),(1,0)\} , \\ (w_i)_{i=1}^7 &:= \{1,\frac{1}{2},\frac{1}{2},1,\frac{1}{2},\frac{1}{2},1\} . \end{aligned}$$

(3 Points)

Exercise 37. (MLS again)

We have already considered the Moving least squares approximation based on a non-negative weight W, a local approximation space $P = \text{span}\langle\varphi_j\rangle_{j=1}^d$, and data $(x_i, f_i)_{i=1}^N$ in exercise 34 but have not shown all related properties. In particular, we have not yet established a relationship between the pointwise minimization of J_x and the linear system $G_x u_x = f_x$.

First we introduce some notions sufficient for G_x to be invertible. For each x define it's neighbourhood of relevant data nodes

$$N(x) := \{x_i : W(x - x_i) > 0\} .$$

We call this neighbourhood P-unisolvent if for all $\phi \in P$ the implication

$$\phi|_{N(x)} = 0 \Rightarrow \phi \equiv= 0$$

holds. In the following, assume that $N(x)$ is P-unisolvent.

a) Show that the necessary condition for π_x minimizing J_x

$$\delta_\pi J_x(\pi_x) = \frac{d}{d\epsilon} J_x(\pi_x + \epsilon\pi)|_{\epsilon=0} \stackrel{!}{=} 0$$

of a vanishing first variation is equivalent to

$$G_x u_x = f_x \ , \qquad \pi_x = \sum_{j=1}^d u_{x,j}\varphi_i$$

when using the basis φ_j of P.

b) Show that assuming $N(x)$ to be P-unisolvent implies that G_x is positive definite. Thus, the unique solution of $G_x u_x = f_x$ is well defined.

c) For the Shepard approximation with $P = \text{span}\langle\varphi_1 \equiv 1\rangle$, find a simplified expression guaranteeing $N(x)$ being P-unisolvent for each $x \in \Omega$.

d) The MLS approximation also admits basis functions. Define the coefficients vector $\alpha_x = (\alpha_{x,j})_{j=1,\ldots,d}$ as the unique solution of the system

$$G_x \alpha_x = P(x) = (\varphi_j(x))_{j=1}^d .$$

Then it holds that

$$\pi(x) = \pi_x(x) = \sum_{i=1}^N f_i \phi_i(x)$$

with basis functions

$$\phi_i(x) := W_i(x) \sum_{j=1}^d \alpha_{x,j}\varphi_j(x_i) .$$

Hint: $\pi_x(x) = \sum_{j=1}^d u_{x,j}\varphi_i(x) = \sum_{j=1}^d u_{x,j}(G_x\alpha_x)_j = \ldots$.

e) Compute ϕ_i for the Shepard approximation and show that $\sum \phi_i \equiv 1$.

(5 Points)

Programming exercise 12. (TrueType fonts)

One commonly used application of Bézier curves and, thus, B-splines is in everyday font rendering. Excluding hinting and rasterization, each glyph (picture of character) of a font in TrueType format is defined by a series of contours given by a composite quadratic Bézier curves.

Each contour is given as a list of points $c_i = (x_i, y_i), i = 1, \dots, N$ with N depending on the particular contour and the following conventions.

1. Contours are closed, i.e. if c_N, c_1 are connected and c_{N-1}, c_N, c_1, c_2 are treated the same way as c_1, c_2, c_3, c_4.
2. Each point c_i is labeled with z_i as *on-curve* ($z_i = 1$) and *off-curve* ($z_i = 0$).
3. Consecutive on-curve points c_i, c_{i+1} with $z_i = z_{i+1} = 1$ are connected by a straight line segment $p(\xi|c_i, c_{i+1})$ or, in other words with the implied off-curve points $c_{i+\frac{1}{2}} = \frac{1}{2}(c_i, c_{i+1})$ on the midpoint giving $p(\xi|c_i, c_{i+\frac{1}{2}}, c_{i+1}) = p(\xi|c_i, c_{i+1})$.
4. The contour description may consist entirely of on-curve points.
5. Triplets beginning and ending at on-curve points and passing through one off-curve points, i.e. c_i, c_{i+1}, c_{i+2} with $z_i = z_{i+2} = 1, z_{i+1} = 0$, are connected with the quadratic Bézier curve $p(\xi|c_i, c_{i+1}, c_{i+2})$.
6. Consecutive off-curve points c_i, c_{i+1} with $z_i = z_{i+1} = 0$ have an implied on-curve point in their middle $c_{i+\frac{1}{2}} = \frac{1}{2}(c_i + c_{i+1})$, $z_{i+\frac{1}{2}} = 1$. This is related to 35e).
7. The contour description may consist entirely of off-curve points.

For example with $N = 4, z_1 = 1, z_2 = 1, z_3 = 0, z_4 = 0$ we have the following outline.

1. There is an implied on-curve point $c_{3\frac{1}{2}} = \frac{1}{2}(c_3 + c_4), z_{3\frac{1}{2}} = 1$.
2. c_1 is connected to c_2 with the straight line $p(\xi|c_1, c_2)$.
3. c_2 is connected to $c_{3\frac{1}{2}}$ with the quadratic Bézier curve $p(\xi|c_2, c_3, c_{3\frac{1}{2}}$.
4. $c_{3\frac{1}{2}}$ is further connected to c_1 with the quadratic Bézier curve $p(\xi|c_{3\frac{1}{2}}, c_4, c_1)$ closing the contour.

On the website find the python file `glyphs.py` including a list `glyphs` with definitions of x_i, y_i, z_i for contours of 3 glyphs from Times New Roman. The file also draws the control polygon of each glyph to exemplify the use of this list.

a) Write a function to add implied on-curve points to consecutive off-curve points of a single contour.

b) Write a function to evaluate the Bézier curve using the Casteljau algorithm given directly in the recursive definition (1).

c) Draw the contours of the 3 TrueType glyphs given in the `glyphs` array.

(4 Points)

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