# Scientific Computing I

Winter Semester 2018/2019
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## Exercise Sheet 1.

Due date: **Tue, 23.10.2018**.

**Exercise 1.** (Green's formulas) (6 Points)

Let $\Omega$ a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. If $u \in \mathcal{C}^1(\Omega)$, we know that

$$\int_\Omega u_{x_i}\,\mathrm{d}x = \int_{\partial\Omega} u\vec{\nu}_i\,\mathrm{d}s, \quad i = 1,\ldots,n, \tag{1}$$

where $\vec{\nu}_i$ is the $i$-th component of the outward pointing normal to the surface of $\Omega$.

Assuming that $u, v \in \mathcal{C}^2(\Omega)$, prove the following identities:

a) $\displaystyle\int_\Omega \Delta u\,\mathrm{d}x = \int_{\partial\Omega} \nabla u \cdot \vec{\nu}\,\mathrm{d}s$

b) $\displaystyle\int_\Omega \nabla u \cdot \nabla v\,\mathrm{d}x = -\int_\Omega u\Delta v\,\mathrm{d}x + \int_{\partial\Omega} u\nabla v \cdot \vec{\nu}\,\mathrm{d}s$

c) $\displaystyle\int_\Omega (u\Delta v - v\Delta u)\,\mathrm{d}x = \int_{\partial\Omega} (u\nabla v \cdot \vec{\nu} - v\nabla u \cdot \vec{\nu})\,\mathrm{d}s$

**Exercise 2.** (PDE classification) (6 Points)

Let $u : \mathbb{R}^3 \times (0,T) \to \mathbb{R}, T > 0$ and $\epsilon > 0$. Investigate if the following partial differential equations are of elliptic, parabolic or hyperbolic type.

a) $\Delta u = a^2 u_{tt}$, $a \in \mathbb{R}$ constant.

b) $\epsilon u_t - \Delta u + au = f$, $a \in \mathbb{R}$ constant.

c) $u_t - \Delta u + \epsilon u_{x_1} - u_{x_2} = f$,

d) $-\Delta u + t(1 - \|\vec{x}\|_2^2)u = f$.

In all the equations the Laplace operator is taken with respect to the spatial variables only, that is, $\Delta u = \sum_{i=1}^3 u_{x_i x_i}$.

**Exercise 3.** (Rotation invariance) (6 Points)

Let $\Omega$ a domain in $\mathbb{R}^n$, $n \geq 1$ and $u : \Omega \to \mathbb{R}$ a function that satisfies the Laplace equation $\Delta u = 0$. Given an orthogonal $n \times n$ matrix $O$, define

$$v(\vec{x}) := u(O\vec{x}) \quad \vec{x} \in \Omega. \tag{2}$$

Show that $\Delta v = 0$.

**Exercise 4.** (Polar coordinates) (6 Points)

a) Show that the Laplace operator $\Delta u = u_{xx} + u_{yy}$ in two dimensions can be written in polar coordinates $(r, \theta)$ as

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{3}$$

b) Let $a$ and $b$ positive constants with $a < b$. Find the solutions of the form $u(r, \theta) = v(r)$ of the equation $u_{xx} + u_{yy} = 1$ in the annular region $a^2 < x^2 + y^2 < b^2$, with $u$ vanishing on both parts of the boundary.